

# Spherically symmetric Anti-de Sitter-like Einstein-Yang-Mills spacetimes

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## Abstract

The conformal field equations are used to discuss the local existence of spherically symmetric solutions to the Einstein-Yang-Mills system which behave asymptotically like the anti-de Sitter spacetime. By using a gauge based on conformally privileged curves we obtain a formulation of the problem in terms of an initial boundary value problem on which a general class of maximally dissipative boundary conditions can be discussed. The relation between these boundary conditions and the notion of mass on asymptotically anti-de Sitter spacetimes is analysed.

**Keywords:** Conformal methods, anti-de Sitter-like, Einstein-Yang-Mills

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## 1 Introduction

The purpose of this article is to give a first step towards the construction, by means of conformal methods, of Einstein-Yang-Mills spacetimes with anti-de Sitter-like boundary conditions. In this first analysis we make the simplifying assumption of spherical symmetry, in order to highlight various issues concerning the choice of boundary conditions in the underlying initial boundary value problem. The construction of more general —i.e. non-symmetric— solutions to the Einstein-Yang-Mills system with anti-de Sitter boundary conditions will be treated elsewhere. Our interest in anti-de Sitter-like spacetimes stems from the numerical results for the spherically symmetric Einstein-scalar field system of [5] which show evidence of turbulent instabilities in arbitrarily small perturbations of the (vacuum) anti-de Sitter spacetime for reflexive boundary conditions —see also e.g. [4, 26] for a further discussion and further references. As the aforementioned work shows, the assumption of spherical symmetry offers a natural starting point to analyse anti de Sitter-like spacetimes, not only from a numerical point of view but also from an analytic perspective —see e.g. [21] the stability of the Schwarzschild-AdS spacetime for the spherically symmetric Einstein-Klein-Gordon system.

The construction of anti-de Sitter-like spacetimes is a much more challenging problem than, say, the construction of de Sitter-like spacetimes. These difficulties stem from the facts that anti-de Sitter-like spacetimes are not globally hyperbolic and that a systematic construction requires the formulation of an *initial boundary value problem* as well as the identification of suitable boundary data. The first systematic construction of anti-de Sitter-like spacetimes by means of an initial boundary value problem has been given in [13]. This seminal work makes use the so-called *conformal field equations* together with a gauge based on the properties of *conformal geodesics*

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to show the *local existence* of a large class of anti-de Sitter-like spacetimes. Besides this analysis for the vacuum case, one should also mention the work in [20, 22] in which the well-posedness of the Einstein-scalar field with anti-de Sitter-like boundary conditions has been analysed.

From our point of view, one of the key lessons of the analysis of [13] is the derivation of the existence of a large class of *maximally dissipative* boundary conditions which ensure the well-posedness of the underlying initial boundary value problem used for the construction of the anti-de Sitter-like spacetimes. In particular, these maximally dissipative boundary conditions allow one to prescribe the conformal class of the conformal boundary. In view of this, it is natural to ponder how crucial the role of the reflexive boundary conditions is in the formation of the instabilities observed in [5]. The set up considered in [13] is probably too general to be used as a starting point for the analysis of the stability/instability of the anti-de Sitter spacetime. From this perspective the assumption of spherical symmetry seems a good way to make inroads. As is well known, the trade-off for considering spherically symmetric configurations in General Relativity is the need of coupling the gravitational field to some matter model in order to obtain non-trivial dynamics. If one intends to use conformal methods for the analysis of this kind of problem, the choice of matter models narrows to those possessing good conformal properties—in particular Yang-Mills fields or the conformally invariant scalar field. The Maxwell field is of no use in spherical symmetry as it leads to the Reissner-Nordström-anti-de Sitter spacetime and trivial dynamics. In the present article we have opted to consider Yang-Mills fields as our matter model. From the point of view of its conformal properties and its coupling to the conformal Einstein field equations, the Yang-Mills field is better behaved than the conformally invariant scalar field. This outweighs the fact that for the Yang-Mills field one has to consider more matter fields. The spherically symmetric Einstein-Yang-Mills equations have been the subject of vast number of studies both analytic—see e.g. [6, 9, 23, 24, 35]—and numeric—see e.g. [2, 3, 7, 31]. It is important to point out that in these studies the gauge freedom available in the specification of the gauge potential has been systematically used to eliminate one of the components of this field, and thus, to obtain a simpler system of equations to be solved. In this article the gauge freedom is used in a different manner to obtain a symmetric hyperbolic evolution system for the components of the gauge potential with nice properties at the conformal boundary. This different strategy in the use of the gauge freedom makes difficult a direct general comparison between the present analysis and other set ups—although a case by case comparison can be certainly obtained if required. Another point that should be stressed is that while most of the analysis of spherically symmetric Einstein-Yang-Mills spacetimes has been carried out for the in which the gauge group is  $SU(2)$ , *our analysis is completely general and makes no assumptions on the gauge group*.

In the present article we pursue a generalisation of the analysis in [13] by combining it with the discussion of the conformal Einstein-Yang-Mills equations given in [12] and that of [25] on the so-called *extended conformal Einstein field equations* for Einstein-Maxwell. Our strategy is to express the extended conformal Einstein field equations in terms of a gauge based on the properties of certain conformally privileged curves, the so-called *conformal curves*, and in turn formulate a boundary value problem including the prescription of certain information on the conformal boundary  $\mathcal{S}$ . The solutions obtained from this initial boundary value problem are, in principle, global in space but local in time. It is well known that the assumption of spherical symmetry leads to a reduction in the number of evolution equations. In addition, this setting exhibits further structural properties which in our opinion deserve a separate treatment. One of the key objectives of the analysis in this article is to identify the boundary data to be prescribed at the conformal boundary in order to ensure the (local) existence of an anti-de Sitter-like Einstein-Yang-Mills spacetime. Our analysis shows that the construction of a Einstein-Yang-Mills spacetime by means of an initial boundary value problem allows to specify a certain combination of the components of the gauge field. More precisely, if  $A_-^{\mathbf{p}}$  and  $A_+^{\mathbf{p}}$  denote the null components of the gauge fields with respect to a frame adapted to the conformal boundary, then

$$A_-^{\mathbf{p}} = c^{\mathbf{p}} A_+^{\mathbf{p}} + q^{\mathbf{p}}, \quad -1 \leq c^{\mathbf{p}} \leq 1 \quad \text{for each } \mathbf{p} \quad (1)$$

with  $q^{\mathbf{p}}$  a collection of smooth functions on the conformal boundary, constitutes a suitable class of *maximally dissipative boundary conditions* ensuring the well-posedness of the underlying initial

boundary value problem. In the above expression  $\mathfrak{p}$  denotes a suitable *gauge index* and no summation is understood on the repeated indices. Our main result can be expressed as follows:

**Theorem.** *Suppose one is given smooth spherically symmetric anti-de Sitter-like initial data for the Einstein-Yang-Mills equations on a 3-dimensional manifold  $\mathcal{S}$  with boundary  $\partial\mathcal{S}$ . Suppose further that the gauge potentials for the Yang-Mills field satisfy the boundary condition (1) on a cylinder  $[0, T] \times \partial\mathcal{S}$  for some  $T > 0$ , and that their initial data on  $\mathcal{S}$  satisfies certain compatibility conditions at the corner  $\{0\} \times \partial\mathcal{S}$ . Then there exists a local-in-time solution to the Einstein-Yang-Mills equations with an anti-de Sitter-like cosmological constant which possesses a conformal completion such that on  $\{0\} \times \mathcal{S}$  it implies the given initial data. Moreover, this solution to the Einstein field equations admits a conformal completion on  $\mathcal{M}_T \equiv [0, T] \times \mathcal{S}$  such that  $\mathcal{I}^+ \equiv [0, T] \times \partial\mathcal{S}$  corresponds to conformal boundary and the boundary conditions (1) are satisfied on  $\mathcal{I}^+$ .*

A detailed discussion of the assumptions made in the above result will be given in the main text —see Section 6, Theorem 1. In particular, in Section 5.3 the meaning of spherical symmetry in the present context will be made precise. The conditions at the corner  $\{0\} \times \partial\mathcal{S}$  (corner conditions) are a hierarchy of compatibility conditions between the boundary data and the initial data ensuring the smoothness of the solution to the initial boundary value problem. Their precise form is discussed in Section 5.4.1. It should also be pointed out that in order to close the problem it is necessary to specify an, in principle, arbitrary gauge source function  $F^{\mathfrak{p}}(x)$  on  $\mathcal{M}_T$  which fixes the value of the divergence of the Yang-Mills gauge potential 1-form. The specification of this gauge source function must be made in a manner which is consistent with boundary conditions. This consistency requirement is part of the corner conditions.

A schematic representation of the result is given in the Penrose diagram of Figure 1. The proof of the above result requires a careful specification of the gauge used to extract an evolution system out of the conformal field equations. In order to avoid a *free boundary problem*, we resort to a conformal gauge based on the properties of a class of curves with nice conformal properties, the so-called conformal curves —see e.g. [25]. A key feature of this gauge is that the conformal factor is known *a priori* since it can be expressed explicitly in terms of initial data. Accordingly, the location of the conformal boundary is explicitly known as well. This feature considerably simplifies the problem and reduces complexity of the PDE theory required to show well-posedness.

Concerning the above result, it should be pointed out that the boundary conditions (1) contain, as a particular case, the *reflective boundary conditions*

$$A_-^{\mathfrak{p}} = A_+^{\mathfrak{p}}.$$

A further interesting property of the setting to be considered concerns the behaviour of the mass. As it will be discussed in more detail in the main text, the spherical symmetry of the setting forces the geometry of the metric intrinsic to the conformal boundary to be conformally flat. That is, the boundary metric is conformally related to the 1+2-dimensional analogues of the Minkowski metric and the metric of the 1+2-dimensional Einstein cylinder universe. For anti-de Sitter-like spacetimes with this property there exists a well defined notion of mass —see [1]. A direct computation shows that for the class of spherically symmetric Einstein-Yang-Mills spacetimes constructed in this article the mass is, in general, not constant along the conformal boundary.

## Outline of the article

In Section 2 we start by discussing background material concerning the Einstein-Yang-Mills field equations in a conformal setting. Section 3 provides an overview of general properties of spherically symmetric anti-de Sitter-like Yang-Mills spacetimes. In particular, this section provides a discussion of the solutions to the constraints implied by the conformal field equations at the conformal boundary of the spacetime and of the behaviour of the mass of the spacetime. Section 4 introduces the concept of conformal curves and studies their properties at  $\mathcal{I}$ . In particular, Lemma 1 proves the existence of conformal curves that remain tangent to the conformal boundary

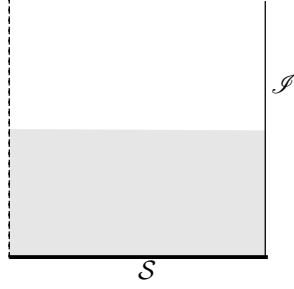


Figure 1: Penrose diagram of the Einstein-Yang-Mills spacetime with anti-de Sitter-like boundary conditions obtained in the present article. The sets  $\mathcal{S}$  and  $\mathcal{J}$  denote, respectively an initial hypersurface and the conformal boundary. The spacetime obtained as a result of our main theorem (indicated by the grey shaded region) is global in space but local in time.

$\mathcal{J}$  for all times. A congruence of conformal curves is used in Section 5 to construct a boundary adapted coordinate system which, in turn, is employed to formulate the initial boundary value problem. Moreover, the congruence is used to suitably fix the conformal gauge freedom so that a hyperbolic reduction of the conformal Einstein-Yang-Mills field equations can be carried out. The equations are analysed in the setting of spherical symmetry in Section 5.3 and in Section 5.4 the boundary conditions for the gauge potential  $A^{\mathfrak{p}}$  are discussed. The main result on the existence of spherically symmetric Einstein-Yang-Mills spacetimes is proven in Section 6, which also presents the technical version of our main theorem. Some concluding remarks are given in Section 7.

## Notation and Conventions

Our signature convention for spacetime (Lorentzian) metrics is  $(+ - - -)$ . As a consequence of this signature convention, the cosmological constant of anti-de Sitter like spacetimes is positive.

In what follows  $a, b, c, \dots$  denote spacetime tensorial indices while  $\mathbf{a}, \mathbf{b}, \mathbf{c}, \dots$  correspond to spacetime frame indices taking the values  $0, \dots, 3$ . For the geometry of a 3-dimensional submanifold the tensorial indices will be denoted by  $i, j, k, \dots$  and frame indices by  $\mathbf{i}, \mathbf{j}, \mathbf{k}$ . The frame indices take values  $\mathbf{0}, \mathbf{1}, \mathbf{2}$  for timelike submanifolds and values  $\mathbf{1}, \mathbf{2}, \mathbf{3}$  for spacelike submanifolds. Part of the analysis will require the use of spinors. In this respect we make use of the general conventions of Penrose & Rindler [28]. In particular,  $A, B, C, \dots$  denote abstract spinorial indices, while  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots$  indicate frame spinorial indices with respect to some specified spin dyad  $\{\delta_{\mathbf{A}}\}$ .

Wherever we consider it preferable for readability, we will suppress tensorial indices and write the corresponding tensor in a bold font. Examples for this are the metric  $\mathbf{g}$ , the frame basis  $\mathbf{e}_{\mathbf{a}}$  and dual cobasis  $\omega^{\mathbf{a}}$ .

Various connections will be used throughout. The connection  $\tilde{\nabla}$  will always denote the Levi-Civita connection of a Lorentzian metric  $\tilde{\mathbf{g}}$  satisfying the Einstein-Yang-Mills field equations — hence, we call it the *physical connection*.  $\nabla$  will denote the Levi-Civita connection of a conformally related *unphysical metric*  $\mathbf{g}$ , while  $\hat{\nabla}$  will denote a Weyl connection of the conformal class  $[\tilde{\mathbf{g}}]$ .

## 2 The conformal Einstein-Yang-Mills equations

In what follows let  $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}}, \mathfrak{G}, \mathbf{F}^{\mathfrak{p}}, \mathbf{A}^{\mathfrak{p}})$  denote a spacetime satisfying the Einstein-Yang-Mills field equations. These field equations provide differential conditions for the spacetime metric  $\tilde{g}_{ab}$  and a set of antisymmetric *gauge fields*  $F^{\mathfrak{p}}_{ab}$  and *gauge potential 1-forms*  $A^{\mathfrak{p}}_a$  where the indices  $\mathfrak{p}, \mathfrak{q}, \dots$

take values in a Lie algebra  $\mathfrak{g}$  of a Lie group  $\mathfrak{G}$ . Explicitly, the field equations are given by

$$\tilde{R}_{ab} - \frac{1}{2}\tilde{R}\tilde{g}_{ab} + \lambda\tilde{g}_{ab} = \tilde{T}_{ab}, \quad (2a)$$

$$\tilde{\nabla}_a A^{\mathfrak{p}}_b - \tilde{\nabla}_b A^{\mathfrak{p}}_a + C^{\mathfrak{p}}_{\mathfrak{qr}} A^{\mathfrak{q}}_a A^{\mathfrak{r}}_b - F^{\mathfrak{p}}_{ab} = 0, \quad (2b)$$

$$\tilde{\nabla}^a F^{\mathfrak{p}}_{ab} + C^{\mathfrak{p}}_{\mathfrak{qr}} A^{\mathfrak{q}a} F^{\mathfrak{r}}_{ab} = 0, \quad (2c)$$

where  $\lambda$  is the cosmological constant and the energy-momentum of the Yang-Mills field is given by

$$\tilde{T}_{ab} = \frac{1}{4}\delta_{\mathfrak{pq}} F^{\mathfrak{p}}_{cd} F^{\mathfrak{q}cd} g_{ab} - \delta_{\mathfrak{pq}} F^{\mathfrak{p}}_{ac} F^{\mathfrak{q}}_b{}^c.$$

In equation (2a)  $\tilde{\nabla}_a$ ,  $\tilde{R}_{ab}$  and  $\tilde{R}$  denote, respectively, the Levi-Civita connection, Ricci tensor and Ricci scalar of the metric  $\tilde{g}_{ab}$  while in equations (2b)-(2c)  $C^{\mathfrak{p}}_{\mathfrak{qr}} = C^{\mathfrak{p}}_{[\mathfrak{qr}]}$  denote the structure constants of the Lie algebra  $\mathfrak{g}$  and  $\delta_{\mathfrak{pq}}$  is the Kronecker delta. Our analysis will make no restriction on the nature of this Lie algebra. It is important to notice that there is a certain gauge freedom in the specification of the Yang-Mills gauge potential 1-form. More precisely,  $A^{\mathfrak{p}}_a$  is determined up to the gradient of a scalar function. In Section 5.2, this gauge freedom will be used to make the divergence of  $A^{\mathfrak{p}}_a$  equal to an arbitrary  $\mathfrak{g}$ -valued function  $\mathcal{F}^{\mathfrak{p}}(x)$  on the spacetime manifold.

The gauge fields  $F^{\mathfrak{p}}_{ab}$  satisfy the *Bianchi identity*

$$\tilde{\nabla}_{[a} F^{\mathfrak{p}}_{bc]} + C^{\mathfrak{p}}_{\mathfrak{qr}} A^{\mathfrak{q}}_{[a} F^{\mathfrak{r}}_{bc]} = 0.$$

By introducing the *Hodge dual*  $F^{\mathfrak{p}*}_{ab}$ , one can rewrite the last equation so that it resembles equation (2c):

$$\tilde{\nabla}^a F^{\mathfrak{p}*}_{ab} + C^{\mathfrak{p}}_{\mathfrak{qr}} A^{\mathfrak{q}a} F^{\mathfrak{r}*}_{ab} = 0. \quad (3)$$

The Yang-Mills equations (2b)-(2c) are conformally invariant. More precisely, suppose we are given the unphysical metric  $g_{ab} = \Xi^2 \tilde{g}_{ab}$  for some conformal factor  $\Xi$  and let  $\nabla_a$  denote the Levi-Civita connection of the metric  $g_{ab}$ . If the fields  $F^{\mathfrak{p}}_{ab}$  and  $A^{\mathfrak{p}}_a$  are solutions to equations (2b)-(2c) for a given  $\tilde{g}_{ab}$ , then they also solve the equations

$$\begin{aligned} \nabla_a A^{\mathfrak{p}}_b - \nabla_b A^{\mathfrak{p}}_a + C^{\mathfrak{p}}_{\mathfrak{qr}} A^{\mathfrak{q}}_a A^{\mathfrak{r}}_b - F^{\mathfrak{p}}_{ab} &= 0, \\ \nabla^a F^{\mathfrak{p}}_{ab} + C^{\mathfrak{p}}_{\mathfrak{qr}} A^{\mathfrak{q}a} F^{\mathfrak{r}}_{ab} &= 0, \end{aligned}$$

For future use it is convenient to define the *unphysical energy-momentum tensor*  $T_{ab}$  as

$$T_{ab} \equiv \Xi^{-2} \tilde{T}_{ab}.$$

Note that the Einstein equations are not conformally invariant and hence  $T_{ab}$  and  $R_{ab}$  are not related by an analogue version of (2a) for the unphysical metric  $g$ . This observation naturally leads to consider the conformal Einstein field equations.

## 2.1 The conformal Einstein field equations with matter

In the present analysis we will make use of a formulation of the conformal field equations expressed in terms of Weyl connections —i.e. a torsion-free connection, not necessarily a Levi-Civita connection, which preserves the conformal structure. This general version of the conformal equations is called the *extended conformal Einstein field equations*. These equations were originally introduced in [12] for the vacuum case and in [25] for the case of matter with trace-free energy-momentum tensor.

For completeness, we briefly review the general setting of the extended conformal field equations for a trace-free energy-momentum tensor. As in the previous section, given a conformal factor we define the *unphysical metric*  $g$  by the relation

$$g = \Xi^2 \tilde{g}.$$

Let  $\{e_a\}$ ,  $a = 0, \dots, 3$  denote a frame field which is  $g$ -orthogonal so that  $g(e_a, e_b) = \eta_{ab}$ , and let  $\{\omega^b\}$  denote its dual cobasis —i.e.  $\langle \omega^b, e_a \rangle = \delta_a^b$ . The connection coefficients  $\Gamma_a^c{}_b = \langle \omega^c, \nabla_a e_b \rangle$

of the Levi-Civita connection  $\nabla$  of  $g$  satisfy the usual metric compatibility condition  $\Gamma_a^b{}_b = 0$ . Given a smooth 1-form  $f$  one can define a Weyl connection  $\hat{\nabla}$  through the relation

$$\hat{\Gamma}_a^c{}_b = \Gamma_a^c{}_b + \delta_a^c f_b + \delta_b^c f_a - \eta_{ab} \eta^{cd} f_d, \quad (4)$$

where  $\hat{\Gamma}_a^c{}_b = \langle \omega^c, \hat{\nabla}_a e_b \rangle$ . In particular, one has that  $f_a = \frac{1}{4} \hat{\Gamma}_a^b{}_b$ . Hence  $\Gamma_a^c{}_b$  can be fully expressed in terms of  $\hat{\Gamma}_a^c{}_b$  using (4).

It is convenient to distinguish between the expression for the components of the Riemann tensor of the connection  $\hat{\nabla}$  in terms of the connection coefficients  $\hat{\Gamma}_a^c{}_b$  (the *geometric curvature*  $\hat{P}^c{}_{dab}$ ) and the expression of the Riemann tensor in terms of the Schouten and Weyl tensors (the *algebraic curvature*  $\hat{\rho}^c{}_{dab}$ ). Explicitly, one has that

$$\begin{aligned} \hat{P}^c{}_{dab} &\equiv e_a(\hat{\Gamma}_b^c{}_d) - e_b(\hat{\Gamma}_a^c{}_d) \\ &\quad + \hat{\Gamma}_f^c{}_d(\hat{\Gamma}_b^f{}_a - \hat{\Gamma}_a^f{}_b) + \hat{\Gamma}_b^f{}_d \hat{\Gamma}_a^c{}_f - \hat{\Gamma}_a^f{}_d \hat{\Gamma}_b^c{}_f, \\ \hat{\rho}^c{}_{dab} &\equiv \Xi d^c{}_{dab} + 2(\delta^c{}_{[a} \hat{L}_{b]d} - \delta^c{}_d \hat{L}_{[ab]} - g_{d[a} \hat{L}_{b]}^c), \end{aligned}$$

where  $d^c{}_{dab} \equiv \Xi^{-1} C^c{}_{dab}$  denotes the components of the rescaled Weyl tensor with respect to the frame  $\{e_a\}$  and  $\hat{L}_{ab}$  those of the Schouten tensor of the connection  $\hat{\nabla}$ .

In order to write down the conformal field equations, it is convenient to define the *geometric zero-quantities*

$$\begin{aligned} \hat{\Sigma}_a^c{}_b e_c &\equiv [e_a, e_b] - (\hat{\Gamma}_a^c{}_b - \hat{\Gamma}_b^c{}_a) e_c, \\ \hat{\Xi}^c{}_{dab} &\equiv \hat{P}^c{}_{dab} - \rho^c{}_{dab}, \\ \hat{\Delta}_{cdb} &\equiv \hat{\nabla}_c \hat{L}_{db} - \hat{\nabla}_d \hat{L}_{cb} - d_a d^a{}_{bcd} - \Xi T_{cdb}, \\ \Lambda_{bcd} &\equiv \nabla_a d^a{}_{bcd} - T_{cdb}, \end{aligned}$$

and the *matter zero-quantities*

$$\begin{aligned} M^p{}_{ab} &\equiv \nabla_a A^p{}_b - \nabla_b A^p{}_a + C^p{}_{qr} A^q{}_a A^r{}_b - F^p{}_{ab}, \\ M^p{}_b &\equiv \nabla^a F^p{}_{ab} + C^p{}_{qr} A^{qa} F^r{}_{ab}, \\ M^{p*}{}_b &\equiv \nabla^a F^{p*}{}_{ab} + C^p{}_{qr} A^{qa} F^{r*}{}_{ab}, \end{aligned}$$

where  $T_{cdb} \equiv \Xi^{-1} \tilde{\nabla}_{[c} \tilde{T}_{d]b}$  denotes the *rescaled Cotton-York tensor*. Expressed it in terms of the unphysical Levi-Civita connection and the unphysical energy-momentum tensor  $T_{ab}$  one has that

$$T_{abc} = \Xi \nabla_{[a} T_{b]c} + 3 \nabla_{[a} \Xi T_{b]c} - g_{c[a} \nabla^e \Xi T_{b]e}. \quad (5)$$

The fields  $f_a$ ,  $d_a$  and  $\Xi$  are related to each other by the constraint

$$d_a = f_a + \nabla_a \Xi.$$

This last expression can be used in formula (5) to eliminate the gradient of the conformal factor.

**Remark.** The geometric zero-quantity  $\hat{\Sigma}_a^c{}_b$  can be viewed as the torsion of the connection  $\hat{\nabla}$ . In the last geometric zero-quantity and in the matter zero-quantities we have used the unphysical Levi-Civita connection  $\nabla$ . This has been done to ease readability of these equations. As mentioned before, the connection coefficients of  $\nabla$  can be expressed entirely in terms of  $\hat{\Gamma}_a^c{}_b$  using (4). The equations in terms of  $\hat{\nabla}$  can be found in [25].

In terms of the above zero-quantities the *extended conformal Einstein-Maxwell field equations* are given by the conditions

$$\hat{\Sigma}_a^c{}_b e_c = 0, \quad \hat{\Xi}^c{}_{dab} = 0, \quad \hat{\Delta}_{cdb} = 0, \quad \Lambda_{bcd} = 0 \quad (6a)$$

$$M^p{}_{ab} = 0, \quad M^p{}_b = 0, \quad M^{p*}{}_b = 0. \quad (6b)$$

The above conformal equations can be read as yielding differential conditions, respectively, for the frame components  $e_a^a$ , the spin coefficients  $\hat{\Gamma}_a^c{}_b$  (including the the components  $f_a$  of the 1-form  $f$ ), the components of the Schouten tensor  $\hat{L}_{ab}$ , the components of the rescaled Weyl tensor  $d^a{}_{bcd}$ , and the collection of matter fields  $F^p{}_{ab}$  and  $A^p{}_a$ .

**Remark.** The conformal equations (6a)-(6b) have to be supplemented with gauge conditions or equations which determine the conformal factor  $\Xi$  and the 1-form  $d$ . This will be discussed in Section 5.1.

### 2.1.1 Spinorial formulation of the equations

The spinorial counterparts of the fields

$$e_a, \quad \hat{\Gamma}_a{}^b{}_c, \quad f_a, \quad \hat{L}_{ab}, \quad d^a{}_{bcd}, \quad d_a, \quad T_{abc},$$

are given, respectively, by the spinor fields

$$e_{AA'}, \quad \hat{\Gamma}_{AA'BC}, \quad f_{AA'}, \quad \hat{L}_{AA'BB'}, \quad \phi_{ABCD}, \quad d_{AA'}, \quad T_{ABCC'}, \quad (7)$$

with

$$\phi_{ABCD} = \phi_{(ABCD)}, \quad T_{ABCC'} = T_{(AB)CC'},$$

The spinorial counterpart of the zero-quantities encoding the conformal Einstein-Yang-Mills field equations is given by

$$\hat{\Sigma}_{AA'}{}^{CC'}{}_{BB'} e_{CC'} \equiv [e_{AA'}, e_{BB'}] - (\hat{\Gamma}_{AA'}{}^{CC'}{}_{BB'} - \hat{\Gamma}_{BB'}{}^{CC'}{}_{AA'}) e_{CC'}, \quad (8a)$$

$$\hat{\Xi}^C{}_{DAA'BB'} \equiv \hat{P}^C{}_{DAA'BB'} - \hat{\rho}^C{}_{DAA'BB'}, \quad (8b)$$

$$\begin{aligned} \hat{\Delta}_{CC'}{}^{DD'}{}_{BB'} &\equiv \hat{\nabla}_{CC'} \hat{L}_{DD'BB'} - \hat{\nabla}_{DD'} \hat{L}_{CC'BB'} \\ &\quad - d^{AA'} (\phi_{ABCD} \epsilon_{A'B'} \epsilon_{C'D'} - \bar{\phi}_{A'B'C'D'} \epsilon_{AB} \epsilon_{CD}) \\ &\quad - \Xi (T_{CDBB'} \epsilon_{C'D'} + \bar{T}_{C'D'B'B} \epsilon_{CD}), \end{aligned} \quad (8c)$$

$$\Lambda_{A'ABC} \equiv \nabla^Q{}_{A'} \phi_{ABCQ} - T_{BCAA'}, \quad (8d)$$

and the reduced geometric and algebraic curvature zero-quantities are given by

$$\begin{aligned} \hat{P}^C{}_{DAA'BB'} &\equiv e_{AA'} (\hat{\Gamma}_{BB'}{}^C{}_D) - e_{BB'} (\hat{\Gamma}_{AA'}{}^C{}_D) \\ &\quad - \hat{\Gamma}_{FB'}{}^C{}_D \hat{\Gamma}_{AA'}{}^F{}_B - \hat{\Gamma}_{BF'}{}^C{}_D \hat{\Gamma}_{AA'}{}^{E'}{}_{B'} + \hat{\Gamma}_{FA'}{}^C{}_D \hat{\Gamma}_{BB'}{}^F{}_A \\ &\quad + \hat{\Gamma}_{AF'}{}^C{}_D \hat{\Gamma}_{BB'}{}^{F'}{}_{A'} + \hat{\Gamma}_{AA'}{}^C{}_E \hat{\Gamma}_{BB'}{}^E{}_D - \hat{\Gamma}_{BB'}{}^C{}_E \hat{\Gamma}_{AA'}{}^E{}_D, \\ \hat{\rho}^C{}_{DAA'BB'} &\equiv \Xi \phi_{CDAB} \epsilon_{A'B'} + L_{DA'BB'} \epsilon_{D'C'} - L_{DB'AA'} \epsilon_{B'C'} \end{aligned}$$

The spinor  $f_{AA'}$  is related to the reduced spin connection coefficients via

$$f_{AA'} = \hat{\Gamma}_{AA'}{}^Q{}_Q.$$

In order to write the Yang-Mills equations in spinorial form it is observed that because of the symmetries of the gauge fields  $F^p{}_{ab}$ , its spinorial counterpart  $F^p{}_{AA'BB'}$  has the decomposition

$$F^p{}_{AA'BB'} = \varphi^p{}_{AB} \epsilon_{A'B'} + \bar{\varphi}^p{}_{A'B'} \epsilon_{AB}, \quad (9)$$

with  $\varphi^p{}_{AB} = \varphi^p{}_{(AB)}$ . In terms of  $\varphi^p{}_{AB}$  the zero quantities take the form

$$\begin{aligned} M^p{}_{AA'BB'} &\equiv \nabla_{AA'} A^p{}_{BB'} - \nabla_{BB'} A^p{}_{AA'} + C^p{}_{q\tau} A^q{}_{AA'} A^\tau{}_{BB'} - \varphi^p{}_{AB} \epsilon_{A'B'} - \bar{\varphi}^p{}_{A'B'} \epsilon_{AB}, \\ M^p{}_{A'A} &\equiv \nabla^Q{}_{A'} \varphi^p{}_{AQ} + C^p{}_{q\tau} A^q{}_{A'} \varphi^\tau{}_{AQ}. \end{aligned}$$

Furthermore, the spinorial counterpart of the energy-momentum tensor  $T_{ab}$  is given by the concise expression

$$T_{AA'BB'} = \delta_{pq} \varphi^p{}_{AB} \bar{\varphi}^q{}_{A'B'}.$$

The spinorial counterpart of the rescaled Cotton-York tensor can be computed from the above expression using the formula

$$T_{ABCC'} = \frac{1}{2} \Xi \nabla_{(A|Q'|} T_{B)}{}^{Q'}{}_{CC'} + \frac{3}{2} \nabla_{(A|Q'|} \Xi T_{B)}{}^{Q'}{}_{CC'} + \nabla^{EE'} \Xi \epsilon_{C(A} T_{B)C'E}{}^{E'}.$$

In terms of the zero-quantities introduced in the previous paragraphs, the spinorial conformal Einstein-Yang-Mills equations are given by the conditions

$$\hat{\Sigma}_{AA'}{}^{CC'}{}_{BB'} e_{CC'} = 0, \quad \hat{\Xi}_{ABCC'}{}^{DD'} = 0, \quad \hat{\Delta}_{AA'BB'}{}^{CC'} = 0, \quad \Lambda_{A'ABC} = 0, \quad (10a)$$

$$M^p{}_{AA'BB'} = 0, \quad M^p{}_{A'A} = 0. \quad (10b)$$

### 3 General properties of spherically symmetric anti-de Sitter-like spacetimes

For completeness, and to motivate the subsequent discussion, we recall a basic proposition concerning the behaviour at the conformal boundary of Einstein-Yang-Mills spacetimes. In what follows, let  $(\tilde{\mathcal{M}}, \tilde{g}, \mathfrak{G}, \mathbf{F}^p, \mathbf{A}^p)$  denote an Einstein-Yang-Mills spacetime —i.e. a spacetime manifold  $\tilde{\mathcal{M}}$ , together with a metric  $\tilde{g}_{ab}$  and  $\mathfrak{g}$ -valued forms  $F^p_{ab}$  and  $A^p_a$  satisfying the Einstein-Yang-Mills equations (2a)-(2c)— and let  $(\mathcal{M}, g, \mathfrak{G}, \mathbf{F}^p, \mathbf{A}^p)$  with  $g = \Xi^2 \tilde{g}$  denote a conformal extension of the physical Einstein-Yang-Mills spacetime. As it is customary, one defines the *conformal boundary*  $\mathcal{I}$  as the set

$$\mathcal{I} \equiv \{p \in \mathcal{M} \mid \Xi = 0\}.$$

One then has that:

**Proposition 1.** *For  $\lambda > 0$ , if the physical energy-momentum tensor of the Yang-Mills field is such that  $T_{abc} = o(\Xi^{-2})$  then  $\mathcal{I}$  is a timelike hypersurface and  $d_{abcd} = O(\Xi^0)$  at  $\mathcal{I}$ .*

The general approach to the proof this results can be found in e.g. [29, 32]. In view of the above result, in what follows, we will say that an Einstein-Yang-Mills spacetime  $(\tilde{\mathcal{M}}, \tilde{g}, \mathfrak{G}, \mathbf{F}^p, \mathbf{A}^p)$  is *anti-de Sitter-like* if  $\lambda > 0$  and there exists a conformal extension  $(\mathcal{M}, g, \mathfrak{G}, \mathbf{F}^p, \mathbf{A}^p)$  with  $g = \Xi^2 \tilde{g}$  such that  $T_{abc} = o(\Xi^{-2})$ .

#### 3.1 Spherically symmetric anti-de Sitter-like spacetimes

An Einstein-Yang-Mills spacetime  $(\tilde{\mathcal{M}}, \tilde{g}, \mathfrak{G}, \mathbf{F}^p, \mathbf{A}^p)$  is said to be *spherically symmetric* if the group  $SO(3)$  acts by isometry on  $(\tilde{\mathcal{M}}, \tilde{g})$  with simply connected, complete, spacelike 2-dimensional orbits —see e.g. [8]— and the  $\mathfrak{g}$ -valued forms  $\mathbf{F}^p$  and  $\mathbf{A}^p$  are invariant under the action of  $SO(3)$ . Given a spherically symmetric spacetime, it is natural to introduce the *quotient manifold*  $\tilde{\mathcal{Q}} \equiv \tilde{\mathcal{M}}/SO(3)$ . The manifold  $\tilde{\mathcal{Q}}$  inherits from  $(\tilde{\mathcal{M}}, \tilde{g})$  a 2-dimensional Lorentzian metric  $\tilde{\gamma}$  —the so-called *quotient metric*. Given a spherically symmetric spacetime, there exists a function  $\tilde{\varrho} : \tilde{\mathcal{Q}} \rightarrow \mathbb{R}$  such that the physical spacetime metric  $\tilde{g}$  can be written in the *warped product* form

$$\tilde{g} = \tilde{\gamma} - \tilde{\varrho}^2 \sigma,$$

where  $\sigma$  is the standard metric of  $\mathbb{S}^2$ .

In what follows, we shall restrict our attention to conformal extensions of spherically symmetric anti-de Sitter-like Einstein-Yang-Mills spacetimes  $(\tilde{\mathcal{M}}, \tilde{g}, \mathfrak{G}, \mathbf{F}^p, \mathbf{A}^p)$  such that the conformal factor  $\Xi$  only depends on the coordinates of the quotient manifold. Under these circumstances, the conformal extension  $(\mathcal{M}, g, \mathfrak{G}, \mathbf{F}^p, \mathbf{A}^p)$  is also spherically symmetric with a quotient manifold  $\mathcal{Q} \equiv \mathcal{M}/SO(3)$  which is a conformal extension of  $\tilde{\mathcal{Q}}$  with  $\gamma = \Xi^2 \tilde{\gamma}$ . The unphysical metric  $g$  is of the form

$$g = \gamma - \varrho^2 \sigma, \quad \varrho : \mathcal{Q} \rightarrow \mathbb{R}. \quad (11)$$

Close to the set of points on  $\mathcal{Q}$  for which  $\Xi = 0$  one can always consider local coordinates  $(t, r)$  such that the conformal boundary  $\mathcal{I}$  is described by the condition  $r = 0$ . The normal to  $\mathcal{I}$  is then given by  $\mathbf{d}r$  with

$$g(\mathbf{d}r, \mathbf{d}r) = \gamma(\mathbf{d}r, \mathbf{d}r) < 0,$$

and  $\mathbf{l}$ , the pull-back of  $g$  to  $\mathcal{I}$ , is of the form

$$\mathbf{l} = A(t) \mathbf{d}t \otimes \mathbf{d}t - B(t) \sigma,$$

with  $A(t)$  and  $B(t)$  two strictly positive functions such that  $B(t) \equiv \varrho(t, 0)$ . The 3-dimensional metric  $\mathbf{l}$  is a Lorentzian metric. Without loss of generality one can redefine the coordinate  $t$  such that  $A(t) = B(t)$  and

$$\mathbf{l} = A(t)(\mathbf{d}t \otimes \mathbf{d}t - \sigma).$$

This 3-dimensional Lorentzian metric can be readily verified to be *conformally flat*.



### 3.1.1 A symmetry adapted frame

The spherical symmetry of the spacetime can be naturally exploited through the choice of a symmetry adapted frame. Let  $\{\mathbf{X}_+, \mathbf{X}_-\}$  denote a basis of  $T\mathbb{S}^2$  consisting of two linearly independent *complex vectors*. The vectors  $\mathbf{X}_+$  and  $\mathbf{X}_-$  can be chosen so that their duals,  $\alpha^+$  and  $\alpha^-$ , satisfy the the relations

$$\langle \alpha^+, \mathbf{X}_+ \rangle = 1, \quad \langle \alpha^-, \mathbf{X}_- \rangle = 1, \quad \langle \alpha^+, \mathbf{X}_- \rangle = 0, \quad \langle \alpha^-, \mathbf{X}_+ \rangle = 0, \quad (12a)$$

$$\sigma = 2(\alpha^+ \otimes \alpha^- + \alpha^- \otimes \alpha^+). \quad (12b)$$

Now, let  $\{\xi_1, \xi_2, \xi_3\}$  denote three (linearly independent) Killing vectors associated to the action of  $SO(3)$  on  $\mathcal{M}$ . The Lie derivatives  $\mathcal{L}_{\xi_j} \mathbf{X}_\pm$  can be computed from the expressions above—in particular, the scaling of the Killing vectors can always be chosen so that

$$\mathcal{L}_{\xi_j} \mathbf{X}_+ = i\mathbf{X}_-, \quad \mathcal{L}_{\xi_j} \mathbf{X}_- = -i\mathbf{X}_+.$$

In what follows, we will consider a basis of  $T\mathcal{Q}$  consisting of two vectors  $\{e_0, e_3\}$  which are orthogonal to each other and normalised with respect to the metric  $\gamma$  in such a manner that

$$\gamma(e_0, e_0) = 1, \quad \gamma(e_3, e_3) = -1,$$

so that in our signature conventions  $e_0$  is timelike and  $e_3$  is spacelike. Letting

$$\omega^0 \equiv \gamma(e_0, \cdot), \quad \omega^3 \equiv \gamma(e_3, \cdot)$$

one has that

$$\gamma = \omega^0 \otimes \omega^0 - \omega^3 \otimes \omega^3.$$

The vectors fields  $\{e_0, e_3\}$  on  $T\mathcal{Q}$  can be extended to vectors on the whole of  $\mathcal{M} = \mathcal{Q} \times \mathbb{S}^2$ , by requiring that

$$[\xi_i, e_0] = 0, \quad [\xi_i, e_3] = 0,$$

so that  $e_0, e_3$  are vectors on  $\mathcal{M}$  which are invariant under the action of  $SO(3)$ . The vectors and  $\{\mathbf{X}_+, \mathbf{X}_-\}$  on  $T\mathbb{S}^2$  are extended to the rest of the spacetime by requiring that

$$[e_0, \mathbf{X}_\pm] = 0, \quad [e_3, \mathbf{X}_\pm] = 0.$$

Finally, defining

$$\mathbf{m}^b \equiv g(\mathbf{m}, \cdot) = \sqrt{2}\varrho\alpha^+, \quad \bar{\mathbf{m}}^b \equiv g(\bar{\mathbf{m}}, \cdot) = \sqrt{2}\varrho\alpha^-,$$

and comparing with the metric (11) one can then write

$$\begin{aligned} g &= \omega_0 \otimes \omega_0 - \omega_3 \otimes \omega_3 - \mathbf{m}^b \otimes \bar{\mathbf{m}}^b - \bar{\mathbf{m}}^b \otimes \mathbf{m}^b, \\ &= \omega_0 \otimes \omega_0 - \omega_3 \otimes \omega_3 - \varrho^2 \sigma. \end{aligned}$$

In particular, one has that  $\mathcal{L}_{\xi_i} g = 0$ , and moreover  $\mathcal{L}_{\xi_i} g^\sharp = 0$ . Using the Cartan structure equations one can compute the connection coefficients and components of the curvature associated to frame basis  $\{e_0, e_3, \mathbf{m}, \bar{\mathbf{m}}\}$ <sup>1</sup>. This will not be further elaborated here.

### 3.1.2 Spherically symmetric Yang-Mills fields

In what follows, we will be interested in Yang-Mills fields which inherit the spherical symmetry of the spacetime. Accordingly, we require that

$$\mathcal{L}_{\xi_j} F^p = 0 \quad (13)$$

---

<sup>1</sup>From the discussion above it follows that this frame can easily be transformed into the typical form of a Newman-Penrose tetrad respecting the spherical symmetry.

As it can be directly verified using the expressions of the previous subsection, the most general form of the field strength consistent with the above requirement can be seen to be given by

$$\mathbf{F}^p = F_{03}^p \omega^0 \wedge \omega^3 + F_{+-}^p \alpha^+ \wedge \alpha^-, \quad (14)$$

with

$$F_{03}^p : \mathcal{Q} \rightarrow \mathbb{R}, \quad F_{+-}^p : \mathcal{Q} \rightarrow \mathbb{C}.$$

Requiring  $\mathbf{F}_{03}^p$  to be real readily implies that  $F_{+-}^p$  must be pure imaginary —that is, one has  $F_{+-}^p = -\overline{F_{+-}^p}$ . Consistent with the above, we will consider gauge potentials of the form

$$\mathbf{A}^p = A_0^p \omega^0 + A_3^p \omega^3 + A_+^p \alpha^+ + A_-^p \alpha^-,$$

with

$$A_0^p, A_3^p : \mathcal{Q} \rightarrow \mathbb{R} \quad A_+^p, A_-^p : \mathcal{Q} \rightarrow \mathbb{C},$$

such that  $\overline{A_+^p} = A_-^p$  in order to ensure the reality of  $\mathbf{A}^p$ . Further conditions on  $\mathbf{A}^p$  can be obtained from the equation  $\mathcal{L}_{\xi_j} M^p_{ab} = 0$  taking into account condition (13). These further conditions will not be required in our subsequent analysis.

### 3.2 The conformal constraint equations at the conformal boundary

In order to obtain a deeper understanding about the structural properties of anti-de Sitter-like spacetimes with matter, it is convenient to consider the *conformal constraint equations* —see e.g. [13] for the vacuum case. In what follows let  $\mathcal{S}$  denote a hypersurface (spacelike or timelike) within the unphysical spacetime  $(\mathcal{M}, g)$ , and let  $\mathbf{n}$  denote its normal. Below, the possibility of  $\mathcal{S}$  being spacelike or timelike are discussed simultaneously by setting  $g(\mathbf{n}, \mathbf{n}) = \epsilon$ , where  $\epsilon = \pm 1$ . If  $\epsilon = 1$ , then the hypersurface is spacelike, while if  $\epsilon = -1$  then it is timelike. As in Section 2 the various conformal fields are expressed in terms of their components with respect to an orthonormal frame  $\{\mathbf{e}_a\}$ . If  $\mathcal{S}$  is spacelike then one naturally sets  $\mathbf{e}_0 = \mathbf{n}$  and one has that the frame indices  $i, j, k, \dots$  take the values  $1, 2, 3$ . By contrast, in the timelike case one sets  $\mathbf{e}_3 = \mathbf{n}$  and the frame indices  $i, j, k, \dots$  take the values  $0, 1, 2$ .

Now, suppose that the frame  $\{\mathbf{e}_a\}$  has been extended off  $\mathcal{S}$  into  $(\mathcal{M}, g)$ . For either of the cases  $\mathbf{e}_0 = \mathbf{n}$  or  $\mathbf{e}_3 = \mathbf{n}$  on  $\mathcal{S}$  extend this notation to  $(\mathcal{M}, g)$  accordingly. Thus,  $\mathbf{n}$  is a vector field in the neighbourhood of  $\mathcal{S}$  and we define

$$\chi_{ij} \equiv g(\nabla_i \mathbf{n}, \mathbf{e}_j) = \Gamma_i^b{}_\perp \eta_{bj} = -\Gamma_i^b{}_j \eta_{b\perp}, \quad \chi \equiv \chi_i^i,$$

where  $\perp$  stands for either  $0$  or  $3$ . Let  $\mathbf{h}$  denote the metric induced by  $g$  on  $\mathcal{S}$ . The components of  $\mathbf{h}$  with respect to the intrinsic frame  $\{\mathbf{e}_i\}$  will be denoted by  $h_{ij} \equiv \eta_{ij}$ . Orthogonal projections of tensors into  $\mathcal{S}$  are given by their components with respect to the interior frame  $\{\mathbf{e}_i\}$ . One writes

$$\Omega \equiv \Xi|_{\mathcal{S}}, \quad \Sigma \equiv \mathbf{n}(\Xi)|_{\mathcal{S}}, \quad K_{ij} \equiv \chi_{ij}|_{\mathcal{S}}, \quad L_i \equiv L_{i\perp}, \quad d_{ij} \equiv d_{i\perp j\perp}, \quad d_{ijk} \equiv d_{i\perp j\perp k}.$$

The components  $d_{ij}$  denote the  $\mathbf{n}$ -electric part of the Weyl tensor while  $d_{ij}^* \equiv -\frac{1}{2} d_{ikl} \epsilon_j{}^{kl}$  correspond to the  $\mathbf{n}$ -magnetic part. In the present formalism  $K_{ij}$  denotes the second fundamental form of the hypersurface  $\mathcal{S}$ . Note, however, that away from  $\mathcal{S}$  the vector field  $\mathbf{n}$  need not be hypersurface orthogonal and hence  $\chi_{ij}$  cannot be interpreted as the second fundamental form of some hypersurface in  $(\mathcal{M}, g)$ . Instead  $\chi_{ij}$  is a more general tensor, which is the reason for our distinction in the notation.

In what follows, the Levi-Civita connection of the metric  $\mathbf{h}$  on  $\mathcal{S}$  will be denoted by  $\mathbf{D}$ . One has that

$$D_i \mathbf{e}_j = \Gamma_i^k{}_j \mathbf{e}_k, \quad \text{on } \mathcal{S},$$

where  $\Gamma_i^k{}_j$  are the components intrinsic to  $\mathcal{S}$  of the connection coefficients of the unphysical spacetime Levi-Civita connection  $\nabla$  computed using the formula

$$\nabla_a \mathbf{e}_b = \Gamma_a{}^c{}_b \mathbf{e}_c.$$

In terms of the fields described in the previous paragraphs, the conformal Einstein constraints at  $\mathcal{S}$  are given for trace-free matter by:

$$D_i D_j \Omega = -\epsilon \Sigma K_{ij} - \Omega L_{ij} + s_{ij} + \frac{1}{2} \Omega^3 T_{ij}, \quad (15a)$$

$$D_i \Sigma = \epsilon K_i^{\mathbf{k}} D_{\mathbf{k}} \Omega - \Omega L_i + \frac{1}{2} \Omega^3 T_{i\perp}, \quad (15b)$$

$$D_i s = -D^{\mathbf{k}} \Omega L_{\mathbf{k}i} - \epsilon \Sigma L_i - \frac{1}{2} \Omega^2 D^j \Omega T_{ij} + \frac{1}{2} \epsilon \Omega^2 \Sigma T_{\perp i}, \quad (15c)$$

$$D_i L_{j\mathbf{k}} - D_j L_{i\mathbf{k}} = D^l \Omega d_{l\mathbf{k}ij} - \epsilon \Sigma d_{\mathbf{k}ij} - K_{i\mathbf{k}} L_j + K_{j\mathbf{k}} L_i + \Omega T_{ij\mathbf{k}}, \quad (15d)$$

$$D_i L_j - D_j L_i = D^l \Omega d_{lij} + K_i^{\mathbf{k}} L_{j\mathbf{k}} - K_j^{\mathbf{k}} L_{i\mathbf{k}} + \Omega T_{ij\perp}, \quad (15e)$$

$$D^{\mathbf{k}} d_{\mathbf{k}ij} = \epsilon (K^{\mathbf{k}}_i d_{j\mathbf{k}} - K^{\mathbf{k}}_j d_{i\mathbf{k}}) + T_{ij\perp}, \quad (15f)$$

$$D^i d_{ij} = K^{i\mathbf{k}} d_{ij\mathbf{k}} + T_{\perp j\perp}, \quad (15g)$$

$$D_j K_{\mathbf{k}i} - D_{\mathbf{k}} K_{ji} = \Omega d_{ij\mathbf{k}} + h_{ij} L_{\mathbf{k}} - h_{i\mathbf{k}} L_j, \quad (15h)$$

$$s_{ij} = \Omega d_{ij} + L_{ij} + \epsilon (K(K_{ij} - \frac{1}{4} K_l^l h_{ij}) - K_{\mathbf{k}i} K_j^{\mathbf{k}} + \frac{1}{4} K_{\mathbf{k}l} K^{kl} h_{ij}), \quad (15i)$$

$$\lambda = 6\Omega s - 3\epsilon \Sigma^2 - 3D_{\mathbf{k}} \Omega D^{\mathbf{k}} \Omega, \quad (15j)$$

where  $s_{ij}$  denotes the components of the Schouten tensor of the intrinsic metric  $\mathbf{h}$ .

As already mentioned in the beginning of this section, the above set-up works both for spacelike ( $\epsilon = 1$ ) and timelike ( $\epsilon = -1$ ) hypersurfaces. In the following we will need to use both cases as we are considering a spacelike initial hypersurface  $\mathcal{S}$  and a timelike conformal boundary  $\mathcal{I}$ . In order to distinguish the two cases and avoid confusion between the two settings we will adopt the following notational conventions. For the spacelike hypersurface  $\mathcal{S}$  we shall adopt the notation as used above. The corresponding notation for the timelike hypersurface  $\mathcal{I}$  will be  $\mathbf{N} = \mathbf{e}_3$  for the normal of  $\mathcal{I}$ ,  $Z = \mathbf{N}(\Xi)$ ,  $\mathbf{l}$  is the intrinsic 3-metric and  $N_{ij} = (\nabla \mathbf{N})_{ij}|_{\mathcal{I}}$  is the extrinsic curvature of  $\mathcal{I}$ .

### 3.3 The conformal constraints at a timelike conformal boundary

The conformal constraint equations discussed in the previous paragraphs acquire a particularly simple form at the conformal boundary of a spacetime. In what follows, it is assumed that both the components of the energy-momentum tensor  $T_{ab}$  and of the rescaled Cotton-York tensor  $T_{abc}$  are regular whenever  $\Xi = 0$ . Furthermore, it is assumed that  $\nabla_a \Xi$  is spacelike so that  $\epsilon = -1$ . As the conformal boundary is a surface of constant  $\Xi$  with normal  $\mathbf{N} = \mathbf{e}_3$ , it follows that the only component of its normal is given by  $Z = \mathbf{N}(\Xi)$  and, consequently,  $D_i \Xi = D_i \Omega = 0$ .

Taking into account the observations raised in the previous paragraph, one has that at a timelike conformal boundary the conformal constraint equations reduce to:

$$\begin{aligned} Z N_{ij} &= -s_{ij}, \\ D_i Z &= 0, \\ D_i s &= Z L_i, \\ D_i L_{j\mathbf{k}} - D_j L_{i\mathbf{k}} &= Z d_{\mathbf{k}ij} - N_{i\mathbf{k}} L_j + N_{j\mathbf{k}} L_i, \\ D_i L_j - D_j L_i &= N_i^{\mathbf{k}} L_{j\mathbf{k}} - N_j^{\mathbf{k}} L_{i\mathbf{k}}, \\ D^{\mathbf{k}} d_{\mathbf{k}ij} &= -N^{\mathbf{k}}_i d_{j\mathbf{k}} + N^{\mathbf{k}}_j d_{i\mathbf{k}}, \\ D^i d_{ij} &= N^{i\mathbf{k}} d_{ij\mathbf{k}} + T_{\perp j\perp}, \\ D_j N_{\mathbf{k}i} - D_{\mathbf{k}} N_{ji} &= l_{ij} L_{\mathbf{k}} - l_{i\mathbf{k}} L_j. \end{aligned}$$

In [13] it has been shown that the solution to the above equations satisfies

$$Z = \sqrt{\lambda/3}, \quad s = \sqrt{\lambda/3} \varkappa, \quad N_{ij} = -\varkappa l_{ij}, \quad L_i = D_i \varkappa, \quad d_{ij}^* = \sqrt{3/\lambda} k_{ij} \quad (16)$$

where  $\varkappa$  is a smooth, gauge dependent real function  $\varkappa : \mathcal{I} \rightarrow \mathbb{R}$  and

$$k_{ij} \equiv -\frac{1}{2} k_{\mathbf{k}l i} \epsilon_j^{\mathbf{k}l}, \quad k_{\mathbf{k}li} \equiv D_{\mathbf{k}} s_{li} - D_l s_{\mathbf{k}i},$$

is the Cotton-York tensor of the 3-metric  $\mathbf{l}$ . In this approach only the components of the electric part of the Weyl tensor need to be solved for. More precisely, one has the equation

$$D^i d_{ij} = 2ZT_{j\perp}, \quad (17)$$

where it has been used that  $T_{\perp j\perp} = 2ZT_{j\perp}$  as a consequence of equation (5) and the fact that  $D_i Z = 0$ .

### 3.3.1 Conformal gauge transformations at the boundary

The form of the solution to the conformal constraint equations given by (16) can be simplified by a suitably choice of the scaling of the unphysical spacetime metric  $\mathbf{g}$ . Under the transition

$$\mathbf{g} \rightarrow \vartheta^2 \mathbf{g}, \quad \Xi \rightarrow \vartheta \Xi, \quad (18)$$

with  $\vartheta \neq 0$  on  $\mathcal{M}$  one has that at the conformal boundary

$$\mathbf{l} \rightarrow (\vartheta|_{\mathcal{J}})^2 \mathbf{l}, \quad s|_{\mathcal{J}} \rightarrow (\vartheta^{-1}s + \vartheta^{-2}\nabla^a \Xi \nabla_a \vartheta)|_{\mathcal{J}} = (\vartheta^{-1}s + \vartheta^{-2}Z\mathbf{N}(\vartheta))|_{\mathcal{J}}.$$

Accordingly, by suitably choosing the values of  $\vartheta$  and  $\mathbf{N}(\vartheta)$  at  $\mathcal{J}$  one can always set  $s = 0$  at the conformal boundary. The expressions in (16) imply that for this scaling one has  $\varkappa = 0$  and hence

$$N_{ij} = 0, \quad L_i = 0.$$

Thus, in this conformal gauge the conformal boundary is extrinsically flat and the spatial components of the Schouten tensor  $L_{ab}$  coincide with the components of the 3-dimensional (intrinsic) Schouten tensor of  $\mathcal{J}$ .

### 3.3.2 Spherical symmetry

In the case of spherically symmetric anti-de Sitter-like spacetimes it has already been shown that the intrinsic metric of the conformal boundary is conformally flat, so that

$$k_{ij} = 0, \quad d_{ij}^* = 0 \quad (\text{spherical symmetry}).$$

Thus for setting considered in this article (16) can be reduced to  $Z = \sqrt{\lambda/3}$  with the remaining conditions vanishing identically.

### 3.4 The Yang-Mills constraints at the conformal boundary

For completeness it is observed that Yang-Mills equations (2c)-(2c) also imply constraints on the conformal boundary. These can be readily seen to be given by :

$$D^i E^p_i + C^p_{qr} A^{qi} E^r_i = 0, \quad (19a)$$

$$D^i B^p_i + C^p_{qr} A^{qi} B^r_i = 0, \quad (19b)$$

where

$$E^p_j \equiv F^p_{\perp j}, \quad B^p_j \equiv F^{p*}_{\perp j},$$

are the *electric and magnetic parts* of  $F^p_{ij}$  with respect to the normal of  $\mathcal{J}$  as defined earlier.

### 3.5 The mass of anti-de Sitter-like spacetimes

As noted above, the intrinsic metric  $\mathbf{l}$  of  $\mathcal{J}$  is conformally flat. Hence, there exists a timelike conformal Killing vector along  $\mathcal{J}$ , i.e. one has  $\xi \in T\mathcal{J}$  such that

$$D_{(i} \xi_{j)} = \frac{1}{3} l_{ij} D^k \xi_k.$$

As discussed in [1], it is therefore possible make use of the conformal constraint equation (17) to write down an integral balance equation over a region of the conformal boundary. A direct computation shows that

$$\begin{aligned} D^i(d_{ij}\xi^j) &= D^i d_{ij}\xi^j + d_{ij}D^i\xi^j \\ &= 2ZT_{j\perp}\xi^j, \end{aligned}$$

where we used (17) and the fact that  $d_{ij}$  is  $\mathbf{l}$ -trace-free. Accordingly, integrating over a region  $\mathcal{R} \subset \mathcal{I}$  bounded by two 2-dimensional surfaces  $\mathcal{C}_1, \mathcal{C}_2 \approx \mathbb{S}^2$  and using the divergence theorem one obtains the balance expression

$$\int_{\mathcal{C}_2} d_{ij}\xi^j \mathbf{d}S^i - \int_{\mathcal{C}_1} d_{ij}\xi^j \mathbf{d}S^i = 2 \int_{\mathcal{R}} ZT_{j\perp}\xi^j \mathbf{d}\mu, \quad (20)$$

where  $\mathbf{d}\mu$  is the volume element of the 3-metric  $\mathbf{l}$  and the area elements  $\mathbf{d}S^i$  are oriented in the direction of the outward pointing normal. The projection  $T_{j\perp}$  is the so-called *Poynting vector*. A calculation with the energy-momentum of the Yang-Mills field shows that

$$T_{i\perp} = -\epsilon_i{}^{jk}\delta_{pq}E^p{}_j B^q{}_k.$$

In *vacuum* the quantity

$$Q[\xi] = \int_{\mathcal{C}} d_{ij}\xi^j \mathbf{d}S^i \quad (21)$$

over an arbitrary section  $\mathcal{C}$  of  $\mathcal{I}$  is conserved. In particular, if  $\xi$  is a timelike conformal Killing vector, then  $Q[\xi]$  can be interpreted as *the mass of the anti-de Sitter-like vacuum spacetime*. More generally, in the presence of a Yang-Mills field, one has that equation (20) describes the change of mass due to the Yang-Mills radiation. The mass will not change if the Poynting vector vanishes.

## 4 Spacetime gauge considerations

The purpose of this section is to discuss the gauge that will be used to obtain an hyperbolic reduction of the conformal Einstein-Yang-Mills equations. This gauge will be based in the properties of a class of conformally privileged curves known as *conformal curves* —see [25]— which, in turn, will be used to propagate coordinates off an initial hypersurface  $\mathcal{S}$ .

### 4.1 Conformal curves

Given a spacetime  $(\tilde{\mathcal{M}}, \tilde{g})$ , a *conformal curve* is a pair  $(\mathbf{x}(\tau), \tilde{\mathbf{b}}(\tau))$  consisting of a curve  $\mathbf{x}(\tau) \in \tilde{\mathcal{M}}$ ,  $\tau \in I \subset \mathbb{R}$  with tangent  $\dot{\mathbf{x}}(\tau) \in T\tilde{\mathcal{M}}$  and a covector  $\tilde{\mathbf{b}}(\tau) \in T^*\tilde{\mathcal{M}}$  along  $\mathbf{x}(\tau)$  satisfying the equations

$$\tilde{\nabla}_{\dot{\mathbf{x}}} \dot{\mathbf{x}} = -2\langle \tilde{\mathbf{b}}, \dot{\mathbf{x}} \rangle \dot{\mathbf{x}} + \tilde{g}(\dot{\mathbf{x}}, \dot{\mathbf{x}}) \tilde{\mathbf{b}}^\sharp, \quad (22a)$$

$$\tilde{\nabla}_{\dot{\mathbf{x}}} \tilde{\mathbf{b}} = \langle \tilde{\mathbf{b}}, \dot{\mathbf{x}} \rangle \tilde{\mathbf{b}} - \frac{1}{2} \tilde{g}^\sharp(\tilde{\mathbf{b}}, \tilde{\mathbf{b}}) \dot{\mathbf{x}}^\flat + \tilde{H}(\dot{\mathbf{x}}, \cdot), \quad (22b)$$

where  $\tilde{H}$  denotes a rank 2 covariant tensor which upon a conformal rescaling  $\mathbf{g} = \Xi^2 \tilde{g}$  transforms as:

$$H_{ab} - \tilde{H}_{ab} = \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b - \frac{1}{2} g^{cd} \Upsilon_c \Upsilon_d g_{ab}, \quad \Upsilon_a \equiv \Xi^{-1} \nabla_a \Xi.$$

This transformation law is *formally identical to that of the Schouten tensor*. The conformal curve equations are supplemented by the following propagation law for a frame  $\{\mathbf{e}_a\}$ :

$$\tilde{\nabla}_{\dot{\mathbf{x}}} \mathbf{e}_a = -\langle \tilde{\mathbf{b}}, \mathbf{e}_a \rangle \dot{\mathbf{x}} - \langle \tilde{\mathbf{b}}, \dot{\mathbf{x}} \rangle \mathbf{e}_a + \tilde{g}(\mathbf{e}_a, \dot{\mathbf{x}}) \tilde{\mathbf{b}}^\sharp. \quad (23)$$

If the 1-form  $\tilde{\mathbf{b}}$  transform as  $\mathbf{b} = \tilde{\mathbf{b}} - \Upsilon$  then it can be verified that

$$\nabla_{\dot{\mathbf{x}}} \dot{\mathbf{x}} = -2\langle \mathbf{b}, \dot{\mathbf{x}} \rangle \dot{\mathbf{x}} + \mathbf{g}(\dot{\mathbf{x}}, \dot{\mathbf{x}}) \mathbf{b}^\sharp, \quad (24a)$$

$$\nabla_{\dot{\mathbf{x}}} \mathbf{b} = \langle \mathbf{b}, \dot{\mathbf{x}} \rangle \mathbf{b} - \frac{1}{2} \mathbf{g}^\sharp(\mathbf{b}, \mathbf{b}) \dot{\mathbf{x}}^\flat + \mathbf{H}(\dot{\mathbf{x}}, \cdot), \quad (24b)$$

$$\nabla_{\dot{\mathbf{x}}} \mathbf{e}_a = -\langle \mathbf{b}, \mathbf{e}_a \rangle \dot{\mathbf{x}} - \langle \mathbf{b}, \dot{\mathbf{x}} \rangle \mathbf{e}_a + \mathbf{g}(\mathbf{e}_a, \dot{\mathbf{x}}) \mathbf{b}^\sharp. \quad (24c)$$

The tensor  $\tilde{\mathbf{H}}$  is, in principle, completely arbitrary. In [25] it is shown that a convenient choice is given by

$$\tilde{\mathbf{H}} = \frac{1}{6}\lambda\tilde{\mathbf{g}} \quad \text{i.e.} \quad \tilde{\mathbf{H}} = \tilde{\mathbf{L}} - \frac{1}{2}\tilde{\mathbf{T}}. \quad (25)$$

**Remark.** A conformal curve is specified by the value of  $\mathbf{x}$ ,  $\dot{\mathbf{x}}$  and  $\tilde{\mathbf{b}}$  at some fiduciary time  $\tau_*$ . The corresponding initial values are denoted in the sequel, respectively, by  $\mathbf{x}_*$ ,  $\dot{\mathbf{x}}_*$ ,  $\tilde{\mathbf{b}}_*$ .

The following result will be fundamental for the construction of our gauge —see [25] for a proof:

**Proposition 2.** *Let  $(\mathbf{x}(\tau), \tilde{\mathbf{b}}(\tau))$  denote a timelike solution curve to the conformal curve equations (22a)-(22b) with the tensor  $\tilde{\mathbf{H}}$  given by (25) on a spacetime  $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$ . If  $\mathbf{g} = \Theta^2\tilde{\mathbf{g}}$  is such that*

$$\mathbf{g}(\dot{\mathbf{x}}, \dot{\mathbf{x}}) = 1, \quad (26)$$

*then the conformal factor  $\Theta$  satisfies*

$$\Theta(\tau) = \Theta_* + \dot{\Theta}_*(\tau - \tau_*) + \frac{1}{2}\ddot{\Theta}_*(\tau - \tau_*)^2, \quad (27)$$

*where the coefficients  $\Theta_* \equiv \Theta(\tau_*)$ ,  $\dot{\Theta}_* \equiv \dot{\Theta}(\tau_*)$  and  $\ddot{\Theta}_* \equiv \ddot{\Theta}(\tau_*)$  are constant along the conformal curve and are subject to the constraints*

$$\dot{\Theta}_* = \langle \tilde{\mathbf{b}}_*, \dot{\mathbf{x}}_* \rangle \Theta_*, \quad \Theta_* \ddot{\Theta}_* = \frac{1}{2}\mathbf{g}^\sharp(\tilde{\mathbf{b}}_*, \tilde{\mathbf{b}}_*) + \frac{1}{6}\lambda.$$

*Furthermore, if  $\{\mathbf{e}_a\}_*$  is an initial  $\mathbf{g}$ -orthogonal frame with  $\mathbf{e}_{0*} = \dot{\mathbf{x}}_*$  which is subsequently propagated along the curve  $\mathbf{x}(\tau)$  according to equation (23) then  $\{\mathbf{e}_a\}$  is  $\mathbf{g}$ -orthogonal for all  $\tau$  and along the conformal curve one has that for all  $\tau$*

$$\Theta\tilde{\mathbf{b}}_0 = \dot{\Theta}, \quad \Theta\tilde{\mathbf{b}}_i = \Theta_*\tilde{\mathbf{b}}_{i*} \quad \text{and} \quad \langle \mathbf{b}, \dot{\mathbf{x}} \rangle = 0,$$

*where  $\tilde{\mathbf{b}}_i \equiv \langle \tilde{\mathbf{b}}, \mathbf{e}_i \rangle$ .*

## 4.2 Conformal curves at the conformal boundary

In view of the purposes of the present article we are particularly interested in the behaviour of conformal curves at the conformal boundary. As it will be seen in the sequel, initial data for the congruence of conformal curves can be chosen in such a way that a conformal curve which is initially tangent to  $\mathcal{S}$  will remain tangent to  $\mathcal{S}$  for all times. For our analysis we will start with a general  $\mathbf{g}$ -orthonormal frame that is adapted to the conformal boundary in the sense that  $\mathbf{N} = \mathbf{e}_3$ . It will be shown that  $\mathbf{e}_3$  is actually Weyl propagated along these boundary intrinsic conformal curves. The discussion in this section is completely general and independent of spherical symmetry.

In what follows, it will be convenient to specify a general orthonormal frame  $\{\mathbf{e}_a\}$  so that  $\mathbf{e}_3$  is normal to  $\mathcal{S}$ . This frame can then be extended to a neighbourhood  $\mathcal{U}$  of  $\mathcal{S}$  by requiring that  $\nabla_{\mathbf{e}_3}\mathbf{e}_3 = 0$ . It follows from this that

$$\Gamma_3^a{}_b = 0 \quad \text{on} \quad \mathcal{U}.$$

If one uses *Gaussian coordinates*  $(x^\mu)$  on  $\mathcal{U}$  based on  $\mathcal{S}$  such that  $\mathcal{S} = \{p \in \mathcal{U} \mid x^3 = 0\}$ , it follows then that

$$e_3^\mu = \delta_3^\mu, \quad e_a^3 = \delta_a^3,$$

where one has written  $\mathbf{e}_a = e_a^\mu \partial_\mu$ . Here and below  $\mu$  refers to components with respect to Gaussian coordinates  $(x^\mu)$ , with  $\mu = 0, \dots, 3$ . We shall use the index  $\alpha$  to denote the restriction to the coordinate values  $0, 1, 2$ .

The tensorial conformal curve equations (24a)-(24b) can be decomposed in components using the boundary adapted frame discussed in the previous paragraph. To this end, one writes

$$\dot{\mathbf{x}} = z^a \mathbf{e}_a, \quad \mathbf{b} = b_a \omega^a.$$

Using this decomposition it is not hard to see that the conformal curve equations split in two groups. Firstly, one has the *normal equations*:

$$\begin{aligned}\dot{x}^3 &= z^a e_a^3 = z^3, \\ \dot{z}^3 &= -\Gamma_a^3 b z^a z^b - 2(b_c z^c) z^3 + (z_c z^c) b^3, \\ \dot{b}_3 &= \Gamma_a^c z^a b_c + (b_c z^c) b_3 - \frac{1}{2}(b_c b^c) z_3 + H_{33} z^3 + H_{i3} z^i.\end{aligned}$$

Secondly, for  $i = 0, 1, 2$  and  $\alpha = 0, 1, 2$  one has the *intrinsic equations*:

$$\begin{aligned}\dot{x}^\alpha &= e_a^\alpha z^a, \\ \dot{z}^i &= -\Gamma_c^i b z^c z^b - 2(b_c z^c) z^i + (z_c z^c) b^i, \\ \dot{b}_i &= \Gamma_b^c z^b b_c + (b_c z^c) b_i - \frac{1}{2}(b_c b^c) z_i + H_{3i} z^3 + H_{ji} z^j.\end{aligned}$$

In order to simplify the analysis of these equations one can exploit the conformal freedom of the setting and choose an element of the conformal class of the intrinsic 3-metric  $\mathbf{l}$  of  $\mathcal{S}$  for which  $s = 0$ . Following the discussion of Section 3.2, this can always be done locally. Under this choice of conformal gauge, the solution of the conformal constraint equations on  $\mathcal{S}$  given in (16) implies that

$$\Gamma_a^3 b = 0, \quad \Gamma_a^c z_3 = 0, \quad L_{3a} = 0.$$

Now, for the class of conformal curves under consideration, the transformation formula of the tensor  $\tilde{\mathbf{H}}$  implies that

$$\mathbf{H} = \mathbf{L} - \frac{1}{2}\Xi^2 \mathbf{T},$$

where  $\mathbf{T}$  denotes the *unphysical energy-momentum tensor*. If the unphysical matter fields are regular at  $\mathcal{S}$ , it follows then that

$$H_{3a} = L_{3a} = 0, \quad H_{ij} = L_{ij} = s_{ij}.$$

That is, the (unphysical) 4-dimensional Schouten tensor  $\mathbf{L}$  is determined by the 3-dimensional Schouten tensor  $\mathbf{s}$  of the intrinsic metric  $\mathbf{l}$  of  $\mathcal{S}$ . From the previous discussion it follows that the normal subset of the conformal curve equations reduces to:

$$\begin{aligned}\dot{x}^3 &= z^3, \\ \dot{z}^3 &= -2(b_c b^c) z^3 + (z_c z^c) b_3, \\ \dot{b}_3 &= (b_c z^c) b_3 - \frac{1}{2}(b_c b^c) z^3.\end{aligned}$$

The key observation is that these equations are homogeneous in the unknowns  $(x^3, z^3, b_3)$ . Thus, by choosing initial data

$$x_\star^3 = 0, \quad \dot{x}_\star^3 = 0, \quad b_{3\star} = 0, \tag{28}$$

one readily obtains a solution

$$x^3(\tau) = 0, \quad z^3(\tau) = 0, \quad b_3(\tau) = 0$$

for later times. Accordingly, conformal curves with initial data given by (28) will remain on  $\mathcal{S}$ . Looking now at the intrinsic part of the conformal curve equations one observes that the equations reduce to

$$\begin{aligned}\dot{x}^\alpha &= z^i e_i^\alpha, \\ \dot{z}^i &= -\Gamma_k^i z^k z^j - 2(b_j z^j) z^i + (z_j z^j) b^i, \\ \dot{b}_i &= \Gamma_j^k z^j b_k + (b_j z^j) b_i - \frac{1}{2}(b_j b^j) z_i + s_{ji} z^j.\end{aligned}$$

These equations are the *conformal geodesic equations* for the conformal structure which is determined by the 3-metric  $\mathbf{l}$  on  $\mathcal{S}$ .

Now let  $\mathbf{v}$  denote a vector satisfying the Weyl propagation equation

$$\nabla_{\dot{\mathbf{x}}} \mathbf{v} = -\langle \mathbf{b}, \mathbf{v} \rangle \dot{\mathbf{x}} - \langle \mathbf{b}, \dot{\mathbf{x}} \rangle \mathbf{v} + \mathbf{g}(\mathbf{v}, \dot{\mathbf{x}}) \mathbf{b}^\sharp$$

along  $\mathcal{J}$ . Making the Ansatz  $\mathbf{v} = \beta \mathbf{e}_3$ , where  $\alpha$  denotes a scalar function on  $\mathcal{J}$  one readily finds that  $z^3(\tau) = 0$  and  $b_3(\tau) = 0$  imply the equation

$$\dot{\beta} = -\langle \mathbf{b}, \dot{\mathbf{x}} \rangle \beta = 0.$$

Thus,  $\beta = \beta_*$  along conformal curves that remain tangent to  $\mathcal{J}$ . Accordingly, if one prescribes at some point of the conformal curve in  $\mathcal{J}$  an orthonormal frame  $\{\mathbf{e}_a\}$  containing a vector which is normal to  $\mathcal{J}$ , one readily finds that the solution to the Weyl propagation equations will be a frame along the conformal curve which contains a vector normal to  $\mathcal{J}$ . Moreover, since Weyl propagation preserves the orthogonality of vectors, it follows that the elements of the frame which are initially tangent to  $\mathcal{J}$  will remain so at later times. In summary, a frame  $\{\mathbf{e}_a\}_*$  that is initially adapted to the boundary will be Weyl propagated into a boundary adapted frame  $\{\mathbf{e}_a\}$ .

The results obtained in the previous paragraphs have been obtained making use of a particular member of the conformal class [1]. It is thus of interest to reformulate them in an arbitrary conformal gauge. To this end one considers on  $\mathcal{M}$ , a conformal factor  $\vartheta > 0$  such that  $\vartheta|_{\mathcal{J}} = 1$  to perform a rescaling as given in (18). In this spirit define

$$\mathbf{g}' \equiv \vartheta^2 \mathbf{g} = (\Xi')^2 \tilde{\mathbf{g}}, \quad \text{with} \quad \Xi' = \vartheta \Xi.$$

This rescaling clearly leaves the boundary metric  $\mathbf{l}$  unchanged in the sense that  $\mathbf{l}' = (\vartheta|_{\mathcal{J}})^2 \mathbf{l}$ . Furthermore, one finds that

$$s'|_{\mathcal{J}} = (\nabla^a \Xi \nabla_a \vartheta)|_{\mathcal{J}} = \sqrt{\lambda/3} \mathbf{e}_3(\vartheta)|_{\mathcal{J}},$$

with  $\mathbf{N} = \mathbf{e}_3$ . Comparing the above expression with (16) suggests defining

$$\varkappa' \equiv \mathbf{e}_3(\vartheta)|_{\mathcal{J}},$$

so that one obtains a general-looking solution to the conformal constraint equations on  $\mathcal{J}$ . Defining a 1-form

$$\mathbf{k} = \vartheta^{-1} \mathbf{d}\vartheta,$$

and taking into account the transformation properties of conformal curves under changes of connections, it follows that  $(\mathbf{x}(\tau), \mathbf{b}'(\tau))$  with  $\mathbf{b}' = \mathbf{b} - \mathbf{k}$  is a solution to the conformal curve equations for the connection  $\nabla' \equiv \nabla + \mathbf{S}(\mathbf{k})$ . From the definition of  $\mathbf{k}$  it follows that  $\nabla'$  is the Levi-Civita connection of the metric  $\mathbf{g}' = \vartheta^2 \mathbf{g}$ . Notice, in particular, that

$$b'_3(\tau)|_{\mathcal{J}} = -k_3(\tau)|_{\mathcal{J}} = -\mathbf{e}_3(\vartheta)|_{\mathcal{J}} = -\varkappa'.$$

The discussion of this section can be summarised as follows:

**Lemma 1.** *Let  $(\mathcal{M}, \mathbf{g})$  be a conformal extension of an anti-de Sitter-like spacetime  $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$  and set  $\tilde{\mathbf{H}} = \frac{1}{6} \lambda \tilde{\mathbf{g}}$ . If  $\gamma$  is a conformal curve which passes through a point  $p \in \mathcal{J}$ , is tangent to  $\mathcal{J}$  at  $p$  and satisfies  $\langle \mathbf{b}, \mathbf{N} \rangle|_p = -\varkappa = -\sqrt{3/\lambda} s$  then  $\gamma$  remains in  $\mathcal{J}$ . Furthermore  $\gamma$  defines a conformal geodesic for the conformal structure of  $\mathcal{J}$  and the Weyl propagation equations in  $(\mathcal{M}, \mathbf{g})$  admit a solution containing a vector field normal to  $\mathcal{J}$ .*

This result is the analogue of Lemma 4.1 in [13] for conformal curves.

## 5 Formulation of an initial boundary value problem

Let  $(\mathcal{M}, \mathbf{g}, \mathbf{F}^{\mathbf{p}}, \mathbf{A}^{\mathbf{p}}, \Xi)$  denote a conformal extension of an oriented and time oriented spherically symmetric anti-de Sitter-like spacetime  $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}}, \tilde{\mathbf{F}}^{\mathbf{p}}, \tilde{\mathbf{A}}^{\mathbf{p}})$  without closed timelike curves. Furthermore, let  $\mathcal{S} \subset \mathcal{M}$  be a smooth, oriented, compact, spacelike hypersurface with boundary  $\partial \mathcal{S}$  that intersects the conformal boundary  $\mathcal{J}$  so that  $\mathcal{S} \cap \mathcal{J} = \partial \mathcal{S}$ . The part of  $\mathcal{J}$  in the future of  $\mathcal{S}$  will be denoted by  $\mathcal{J}^+$ . For convenience of the discussion, it will be assumed that the causal future  $J^+(\mathcal{S})$  coincides with  $D^+(\mathcal{S} \cup \mathcal{J}^+)$ , the future domain of dependence of the set  $\mathcal{S} \cup \mathcal{J}^+$



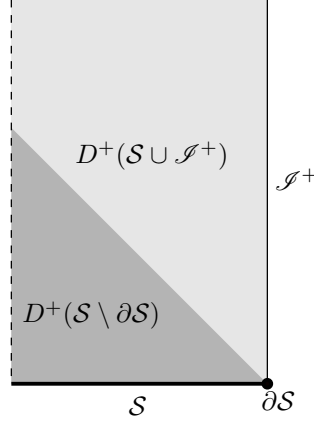


Figure 2: Penrose diagram of the set up for the construction of anti de Sitter-like spacetimes as described in the main text. Initial data prescribed on  $\mathcal{S} \setminus \partial\mathcal{S}$  allows to recover the dark shaded region  $D^+(\mathcal{S} \setminus \partial\mathcal{S})$ . In order to recover  $D^+(\mathcal{S} \cup \mathcal{J}^+)$  it is necessary to prescribe boundary data on  $\mathcal{J}^+$ . Notice that  $D^+(\mathcal{S} \cup \mathcal{J}^+) = J^+(\mathcal{S})$ .

—a schematic depiction of this setting can be seen in Figure 2. In the following we will deal with results that are local in time. In particular, we will assume that we work with a set  $\mathcal{N} \subset \mathcal{M}$  of the form  $\mathcal{N} \approx [0, 1] \times \mathcal{S} \subset D^+(\mathcal{S} \cup \mathcal{J}^+)$  so that  $\mathcal{J}^+ \cap \mathcal{N} \approx [0, 1] \times \partial\mathcal{S}$ . For the definition of the above causal notions and some of their properties, see [34].

In what follows we will address the following

**Question:** *What data does one need to specify on  $\mathcal{S} \cup \mathcal{J}^+$  to reconstruct (up to diffeomorphisms) the anti-de Sitter-like Einstein-Yang-Mills spacetime  $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}}, \tilde{\mathbf{F}}^{\mathbf{p}}, \tilde{\mathbf{A}}^{\mathbf{p}})$  in a neighbourhood  $\mathcal{U}$  of  $\mathcal{S}$ , where  $\mathcal{U} \subset J^+(\mathcal{S})$ ?*

It is a consequence of the standard Cauchy problem in General Relativity that the solution to the Einstein-Yang-Mills field equations on the domain of dependence of  $\tilde{\mathcal{S}} \equiv \mathcal{S} \setminus \partial\mathcal{S}$  is determined in a unique manner, up to diffeomorphisms, by a collection of tensors  $(\tilde{\mathbf{h}}, \tilde{\mathbf{K}}, \tilde{\mathbf{E}}^{\mathbf{p}})$  satisfying the Einstein-Yang-Mills constraint equations on  $\tilde{\mathcal{S}}$ . In order to be able to recover  $J^+(\mathcal{S}) \setminus D^+(\tilde{\mathcal{S}})$  one needs to prescribe suitable initial data on the conformal boundary  $\mathcal{J}$ . Identifying this initial data requires a suitable gauge — that is, a choice of conformal scaling, orthonormal frame and coordinate system — in which the problem can be analysed. In particular, one needs to specify a gauge near  $\mathcal{J}$ . *Following the analysis of the conformal curves in the previous sections of this article, we will fix the gauge choice using a congruence of conformal curves.*

## 5.1 Fixing the gauge

Following the conventions of Section 3.2, set  $\Omega \equiv \Xi|_{\mathcal{S}}$  below. In order to simplify the subsequent discussion, it is assumed that the initial spacelike hypersurface  $\mathcal{S}$  has been chosen so that  $\mathcal{S}$  and  $\mathcal{J}$  meet orthogonally —that is, on  $\partial\mathcal{S}$  the unit normal  $\mathbf{n} = \mathbf{e}_0$  to  $\mathcal{S}$  is tangent to  $\mathcal{J}$  and one has that  $\Sigma = \mathbf{n}(\Xi) = \langle \mathbf{d}\Xi, \mathbf{n} \rangle = 0$  on  $\partial\mathcal{S}$ . Under these circumstances, the conformal factor  $\Xi$  can be chosen such that

$$\mathbf{n}(\Xi) = \Sigma = 0, \quad \text{on} \quad \mathcal{S}.$$

Moreover, recalling that at the conformal boundary  $s$  can be made to vanish by a convenient choice of conformal gauge, it is assumed that

$$s = 0, \quad \text{on} \quad \partial\mathcal{S}.$$

Now, each  $p \in \mathcal{S}$  is assumed to be the starting point of a future directed conformal curve  $(\mathbf{x}(\tau), \mathbf{b}(\tau))$  and an associated Weyl propagated frame  $\{\mathbf{e}_a\}$ . The parametrisation of the curves is naturally chosen so that  $\tau = 0$  on  $\mathcal{S}$ . For points  $p \in \tilde{\mathcal{S}}$  the initial data for these curves is set in terms of  $\tilde{\mathbf{g}}$  and its Levi-Civita connection  $\tilde{\nabla}$  by the conditions:

- (i)  $\dot{\mathbf{x}}$  is future directed, orthogonal to  $\tilde{\mathcal{S}}$  and  $\tilde{g}(\dot{\mathbf{x}}, \dot{\mathbf{x}}) = \Omega^{-2}$ ,
- (ii)  $\mathbf{b}_\perp = \Omega^{-1} \mathbf{d}\Omega$  and  $\langle \mathbf{b}, \dot{\mathbf{x}} \rangle = 0$  (in line with  $\Sigma = 0$ ).
- (iii)  $\mathbf{e}_0 = \dot{\mathbf{x}}$  and  $\tilde{g}(\mathbf{e}_a, \mathbf{e}_b) = \Omega^{-2} \eta_{ab}$ .

**Remark:** Once the above conditions have been used to fix the gauge both the set covered by the points of a conformal curve as well as its parameter  $\tau$  are independent of the remaining freedom of prescribing the frame  $\{\mathbf{e}_a\}$  on  $\mathcal{S}$ .

On suitable neighbourhoods  $\mathcal{W} \subset J^+(\mathcal{S})$  of  $\mathcal{S}$  such that their intersection with conformal curves is connected one has that the curves  $\mathbf{x}(\tau)$  define a smooth timelike congruence in  $\mathcal{W}$ ,  $\{\mathbf{e}_a\}$  a smooth frame field and  $\mathbf{b}$  a smooth 1-form. The conformal curves can be used to construct a conformal Gaussian coordinate system  $x = (x^\mu)$  on  $\mathcal{W}$ . For this construction choose a set of local coordinates  $(x^\alpha)$  on  $\mathcal{S}$  and extend it off  $\mathcal{S}$  by keeping  $(x^\alpha)$  constant along individual conformal curves. Finally set  $x^0 = \tau$ , the conformal parameter along the conformal curves.

On  $\mathcal{W}$  the coefficients  $e_a{}^\mu = \langle \mathbf{d}x^\mu, \mathbf{e}_a \rangle$  of  $\mathbf{e}_a$  with respect to the Gaussian coordinates satisfy  $e_0{}^\mu = \delta_0{}^\mu$ . Notice, however, that in general  $e_a{}^0 = 0$  holds only on  $\mathcal{S}$ . The conformal factor  $\Xi$  is then fixed on  $\mathcal{W}$  by requiring  $g(\mathbf{e}_a, \mathbf{e}_b) = \eta_{ab}$  so that

$$\Xi = \Theta.$$

with  $\Theta$  given in (27).

In order to include the conformal curves on the conformal boundary in the discussion one has to make use of the unphysical metric  $g$  and its Levi-Civita connection  $\nabla$ . In terms of  $g$  and  $\nabla$ , the conformal curves are represented by the pair  $(\mathbf{x}(\tau), \mathbf{f}(\tau))$  with  $\mathbf{f} = \mathbf{b} - \Theta^{-1} \mathbf{d}\Theta$ . Accordingly, one has that

$$\mathbf{f} = 0, \quad \text{on} \quad \mathcal{S}.$$

Now recall that on  $\partial\mathcal{S}$  we can fix  $b_{3*} = 0$  in agreement with our assumption of  $s = 0$  there. Hence by Lemma 1 one has that conformal curves which start on  $\partial\mathcal{S}$  remain on  $\mathcal{S}$ . As  $s = 0$  on  $\partial\mathcal{S}$  one can write

$$s = \Omega \varsigma_*, \quad \text{on} \quad \mathcal{S}$$

where  $\varsigma_*$  is a smooth function on  $\mathcal{S}$  with  $\varsigma_* \neq 0$  on  $\partial\mathcal{S}$ . It follows then that  $\Theta_* = \Omega$ ,  $\dot{\Theta}_* = 0$ ,  $\ddot{\Theta} = \ddot{\Theta}_* = \Omega \varsigma_*$  and hence

$$\Theta = \Omega \left(1 - \frac{1}{2} \varsigma_* \tau^2\right). \quad (29)$$

Moreover, we have

$$d_0 = \dot{\Theta}, \quad d_i = d_{i*} = \mathbf{e}_i(\Omega)_* \equiv (e_i{}^\alpha \partial_\alpha \Omega)_*$$

where the functions  $\Omega$ ,  $\varsigma_*$  and  $\mathbf{e}_i(\Omega)_*$  defined initially on  $\mathcal{S}$  are extended to  $\mathcal{W}$  so that they are constant along individual conformal curves. The gauge system described in the previous paragraphs will be known as a *boundary adapted gauge*.

## 5.2 Hyperbolic reduction of the Einstein-Yang-Mills equations

The procedure of deducing a symmetric hyperbolic evolution system from the extended conformal field equations with the help of a conformal gauge system based on conformal curves has been discussed in [25]. This hyperbolic reduction procedure is conveniently implemented by means of a space spinor formalism, the details of which can be found in [12, 14] and are thus not presented here.

Let  $\tau_{AA'}$  denote the spinorial counterpart of the tangent vector to the conformal curves with normalisation  $\tau_{AA'} \tau^{AA'} = 2$ . In what follows, a spin dyad  $\{\epsilon_A\}$  is chosen such that

$$\tau_{AA'} \equiv \tau_{AA'} \epsilon_A{}^A \bar{\epsilon}_{A'}{}^{A'} = \delta_A{}^0 \delta_{A'}{}^{0'} + \delta_A{}^1 \delta_{A'}{}^{1'}.$$

The spinor  $\tau_{AA'}$  is then used to introduce a space spinor formalism which allows one to express all the unknowns in the conformal Einstein-Yang-Mills equations in terms of spinors without primed indices. In particular, one defines

$$\begin{aligned} e_{AB} &\equiv \tau_B^{A'} e_{AA'}, & \hat{\Gamma}_{ABCD} &\equiv \tau_B^{A'} \hat{\Gamma}_{AA'CD}, & f_{AB} &\equiv \tau_B^{A'} f_{AA'}, \\ \hat{L}_{ABCD} &\equiv \tau_B^{A'} \tau_D^{C'} \hat{L}_{AA'CC'}, & A^p_{AB} &\equiv \tau_B^{A'} A^p_{AA'}. \end{aligned}$$

The above fields admit the decomposition [14]

$$\begin{aligned} e_{AB} &= \frac{1}{2} \epsilon_{AB} e_Q^Q + e_{(AB)}, \\ \hat{\Gamma}_{ABCD} &= \frac{1}{\sqrt{2}} (\xi_{ABCD} - \chi_{ABCD}) + \epsilon_{AC} f_{DB}, \\ f_{AB} &= \frac{1}{2} \epsilon_{AB} f_Q^Q + f_{(AB)}, \\ \hat{L}_{ABCD} &= \frac{1}{2} \epsilon_{AB} \hat{L}_Q^Q{}_{CD} + \frac{1}{2} \epsilon_{CD} \hat{L}_{ABQ}^Q + \hat{L}_{(AB)(CD)}. \end{aligned}$$

On the initial hypersurface  $\mathcal{S}$ , the fields  $\xi_{ABCD}$  and  $\chi_{ABCD}$  are associated to the intrinsic (Levi-Civita) connection and the extrinsic curvature of  $\mathcal{S}$ . Since our congruence of conformal curves is not necessarily hypersurface orthogonal this interpretation does not hold off  $\mathcal{S}$ .

The gauge conditions associated to the congruence of conformal curves can be expressed in terms of space-spinor objects as

$$e_Q^Q = \sqrt{2} \partial_\tau, \quad \hat{\Gamma}_Q^Q{}_{CD} = 0, \quad f_Q^Q = 0, \quad \hat{L}_Q^Q{}_{CD} = \frac{1}{2} \delta_{pq} \phi_{QC}{}^p \phi^\dagger{}^Q{}_D{}^q. \quad (30)$$

Hence, writing  $e_{AB} = e_{AB}{}^\mu \partial_\mu$  with  $\mu = 0, 3, \pm$  one has, in particular, that

$$e_{AB} = \sqrt{2} \epsilon_{AB} \partial_\tau + (e_{(AB)}^0 \partial_\tau + e_{(AB)}^3 \partial_3 + e_{(AB)}^+ X_+ + e_{(AB)}^- X_-),$$

where  $X_\pm$  are the vectors on  $T\mathbb{S}^2$  introduced in Section 3.1.1.

In the spirit of the space-spinor formalism, it is convenient to express the conformal Einstein-Yang-Mills equations (10a)-(10b) in terms of the following (equivalent) space-spinor zero-quantities

$$\begin{aligned} \hat{\Sigma}_{ABCD} &= 0, & \hat{\Xi}_{AB CDEF} &= 0, & \hat{\Delta}_{AB CDEF} &= 0, & \Lambda_{ABCD} &= 0, \\ M_{ABCD}^p &= 0, & M_{AB}^p &= 0, \end{aligned}$$

which are obtained by suitably contracting the original spinorial zero quantities with the spinor  $\tau_{AA'}$  —e.g.  $\hat{\Sigma}_{ABCD} \equiv \tau_B^{A'} \tau_D^{C'} \hat{\Sigma}_{AA'CC'}$ . Taking into account the gauge conditions (30), the evolution equations are obtained from

$$\hat{\Sigma}_Q^Q{}_{CD} = 0, \quad \hat{\Xi}_Q^Q{}_{CDEF} = 0, \quad \hat{\Delta}_Q^Q{}_{CDEF} = 0, \quad \Lambda_{(ABCD)} = 0, \quad (31a)$$

$$M_{(AB)}^p = 0, \quad M_{AQB}^p{}^Q + \epsilon_{AB} (\nabla^{PQ} A_{PQ}^p - \mathcal{F}^p) = 0, \quad (31b)$$

where  $\mathcal{F}^p = \mathcal{F}^p(x)$  is an arbitrary gauge source function expressing the freedom available in the specification of the potential  $A_a^p$  —see e.g. [11, 12]. This gauge source function allows one to set the divergence of the Yang-Mills potential equal to any arbitrary function.

**Remark 1.** In vacuum, the conditions (31a)-(31b) give rise to a symmetric hyperbolic system for the various (geometric) conformal fields —see e.g. [14]. In the presence of matter this is no longer the case as the Cotton-York spinor  $T_{ABCD} \equiv \tau_D^{C'} T_{ABCC'}$  appearing in the zero-quantities  $\hat{\Delta}_{AB CDEF}$  and  $\Lambda_{ABCD}$  contains derivatives of the spinor field  $\varphi_{AB}^p$  which cannot be eliminated by, say, using the matter field equations. In order to get around this complication one introduces the derivatives  $\hat{\nabla}_{AA'} \varphi^p{}_{BC}$  as further unknowns in the evolution system —see [13, 25]. Remarkably, as it will be seen in the sequel, *under a suitable Ansatz for spherical symmetry all the non-vanishing derivatives of the field  $\varphi_{AB}^p$  can be obtained from the condition  $M_{AB}^p = 0$ .*

**Remark 2.** The evolution condition  $\Lambda_{(ABCD)} = 0$  leads to the so-called *standard evolution system* for the independent components of the Weyl spinor  $\phi_{ABCD}$ . When considering a boundary value problem, the so-called *boundary adapted system* presented in [14] provides a more convenient

evolution system. Under a suitable Ansatz for spherical symmetry (see below) the two evolution systems coincide.

In order to show that a solution to the evolution conditions (31a)-(31b) implies a solution to the conformal Einstein-Yang-Mills equations (10a)-(10b), one constructs a subsidiary evolution system for the geometric and matter zero-quantities. In particular, this subsidiary system is homogeneous in the zero-quantities. This lengthy procedure has been discussed in [12, 13, 25]. Due to the homogeneity of the subsidiary system, the vanishing of the zero-quantities on the initial hypersurface  $\mathcal{S}$  implies that the zero-quantities must vanish elsewhere. It can be readily verified that a solution to the full conformal Einstein-Yang-Mills equations implies a solution to the Einstein-Yang-Mills equations with anti-de Sitter-like cosmological constant on the set where the conformal factor is non-vanishing —see e.g. [11].

### 5.3 The evolution equations in spherical symmetry

The evolution equations (31a)-(31b) can be reformulated in terms of scalar equations by decomposing the spinorial fields into irreducible components. Taking into account the symmetries of the fields and the gauge conditions (30) one can write

$$\begin{aligned}
e_{(AB)}^\mu &= e_x^\mu x_{AB} + e_y^\mu y_{AB} + e_z^\mu z_{AB}, \quad \mu = 0, 3, \pm \\
f_{(AB)} &= f_x x_{AB} + f_y y_{AB} + f_z z_{AB}, \\
\xi_{ABCD} &= \xi_0 \epsilon_{ABCD}^0 + \xi_1 \epsilon_{ABCD}^1 + \xi_2 \epsilon_{ABCD}^2 + \xi_3 \epsilon_{ABCD}^3 + \xi_4 \epsilon_{ABCD}^4 \\
&\quad + \xi_{\epsilon x} (\epsilon_{AC} x_{BD} + \epsilon_{BD} x_{AC}) + \xi_{\epsilon y} (\epsilon_{AC} y_{BD} + \epsilon_{BD} y_{AC}) + \xi_{\epsilon z} (\epsilon_{AC} z_{BD} + \epsilon_{BD} z_{AC}), \\
\chi_{ABCD} &= \chi_0 \epsilon_{ABCD}^0 + \chi_1 \epsilon_{ABCD}^1 + \chi_2 \epsilon_{ABCD}^2 + \chi_3 \epsilon_{ABCD}^3 + \chi_4 \epsilon_{ABCD}^4 + \chi_h h_{ABCD} \\
&\quad + \chi_{\epsilon x} (\epsilon_{AC} x_{BD} + \epsilon_{BD} x_{AC}) + \chi_{\epsilon y} (\epsilon_{AC} y_{BD} + \epsilon_{BD} y_{AC}) + \chi_{\epsilon z} (\epsilon_{AC} z_{BD} + \epsilon_{BD} z_{AC}), \\
\hat{L}_{(AB)CD} &= \theta_0 \epsilon_{ABCD}^0 + \theta_1 \epsilon_{ABCD}^1 + \theta_2 \epsilon_{ABCD}^2 + \theta_3 \epsilon_{ABCD}^3 + \theta_4 \epsilon_{ABCD}^4 + \theta_h h_{ABCD} \\
&\quad + \theta_{\epsilon y} (\epsilon_{AC} y_{BD} + \epsilon_{BD} y_{AC}) + \theta_{\epsilon z} (\epsilon_{AC} z_{BD} + \epsilon_{BD} z_{AC}) \\
&\quad + \frac{1}{\sqrt{2}} \epsilon_{CD} (\theta_x x_{AB} + \theta_y y_{AB} + \theta_z z_{AB}), \\
\phi_{ABCD} &= \phi_0 \epsilon_{ABCD}^0 + \phi_1 \epsilon_{ABCD}^1 + \phi_2 \epsilon_{ABCD}^2 + \phi_3 \epsilon_{ABCD}^3 + \phi_4 \epsilon_{ABCD}^4, \\
\varphi^p_{AB} &= \varphi_x^p x_{AB} + \varphi_y^p y_{AB} + \varphi_z^p z_{AB}, \\
A^p_{AB} &= A^p \epsilon_{AB} + A_x^p x_{AB} + A_y^p y_{AB} + A_z^p z_{AB},
\end{aligned}$$

where the basic irreducible spinors

$$\begin{aligned}
&x_{AB}, \quad y_{AB}, \quad z_{AB}, \quad h_{ABCD}, \\
&\epsilon_{ABCD}^0, \quad \epsilon_{ABCD}^1, \quad \epsilon_{ABCD}^2, \quad \epsilon_{ABCD}^3, \quad \epsilon_{ABCD}^4
\end{aligned}$$

are defined in the Appendix. For future reference it is convenient to group the coefficients in the above Ansatz in the vector unknown

$$\mathbf{u} \equiv (e_x^0, e_y^0, \dots, A_z^p).$$

In the case of a spherically symmetric spacetime —see the discussion of Section 3.1— one can introduce a spin dyad adapted to the symmetry of the spacetime. A suitable Ansatz for spherical symmetry in the present setting is given by

$$e_{(AB)}^0 = e_x^0 x_{AB}, \quad e_{AB}^3 = e_x^3 x_{AB}, \quad e_{AB}^+ = e_z^+ z_{AB}, \quad e_{AB}^- = e_y^- y_{AB}, \quad (32a)$$

$$f_{AB} = f_x x_{AB}, \quad \xi_{ABCD} = \frac{1}{\sqrt{2}} \xi_{\epsilon x} (\epsilon_{AC} x_{BD} + \epsilon_{BD} x_{AC}), \quad (32b)$$

$$\chi_{ABCD} = \chi_2 \epsilon_{ABCD}^2 + \frac{1}{3} \chi_h h_{ABCD}, \quad (32c)$$

$$\hat{L}_{(AB)CD} = \theta_2 \epsilon_{ABCD}^2 + \frac{1}{3} \theta_h h_{ABCD} + \frac{1}{\sqrt{2}} \theta_x \epsilon_{CD} x_{AB}, \quad (32d)$$

$$\phi_{ABCD} = \phi_2 \epsilon_{ABCD}^2, \quad \varphi_{AB}^p = \varphi_x^p x_{AB}, \quad (32e)$$

$$A_{AB}^p = A^p \epsilon_{AB} + A_x^p x_{AB} + A_y^p y_{AB} + A_z^p z_{AB} \quad (32f)$$

where all the coefficients in the above Ansatz depend only on the coordinates of the quotient space  $\mathcal{Q}$  and except for  $e^+$ ,  $e^-$  and  $\varphi_x^p$  are real. Moreover, save for  $e^+$  and  $e^-$  all the coefficients have spin-weight 0. For more details on the motivations behind the spherical symmetric Ansatz (32a)-(32f), we refer the reader to [14, 33]. To see that the Ansatz is consistent with the warp product metric (11) one can compute  $\mathbf{g}$  in terms of the frame coefficients  $e_x^0, e_x^1, e_z^+, e_y^-$  using the formula

$$\mathbf{g} = \epsilon_{AB}\epsilon_{A'B'}\omega^{AA'} \otimes \omega^{BB'},$$

where  $\{\omega^{AA'}\}$  denotes the dual cobasis to  $\{e_{BB'}\}$ —that is, one has  $\langle \omega^{AA'}, e_{BB'} \rangle = \delta_A^B \delta_{A'}^{B'}$ . Defining  $\omega^{AB} \equiv -\omega^{AQ'}\tau_{Q'}^B$ , one has the decomposition

$$\omega^{AB} = \frac{1}{2}\epsilon^{AB}\omega_Q^Q + \omega^{(AB)}.$$

Some computations yield the relations

$$\begin{aligned}\omega_P^P &= \sqrt{2} \left( d\tau - \frac{e_x^0}{e_x^1} dx^3 \right), \\ \omega^{(AB)} &= -\frac{1}{e_x^1} x^{AB} dx^3 - \frac{2}{e_z^+} y^{AB} \alpha^+ - \frac{2}{e_y^-} z^{AB} \alpha^-.\end{aligned}$$

Taking into account the above expressions one further finds that

$$\mathbf{g} = d\tau \otimes d\tau - \frac{e_x^0}{e_x^1} (d\tau \otimes dx^3 + dx^3 \otimes d\tau) - \left( \frac{1}{(e_x^1)^2} - \left( \frac{e_x^0}{e_x^1} \right)^2 \right) dx^3 \otimes dx^3 - \frac{1}{e_z^+ e_y^-} \sigma,$$

which is in the required warped-product form. An expression for the gauge field  $\mathbf{F}^p$  can be computed in a similar manner from

$$\mathbf{F}^p = F_{AA'BB'}^p \omega^{AA'} \otimes \omega^{BB'},$$

which, recalling the split (9) of the spinorial counterpart of  $F_{AA'BB'}^p$  gives

$$F_{AA'BB'}^p = \varphi_x^p x^{AB} \epsilon_{A'B'} + \bar{\varphi}_x^p \bar{x}_{A'B'} \epsilon_{AB}$$

so that

$$\mathbf{F}^p = \frac{1}{\sqrt{2}}(\varphi_x^p + \bar{\varphi}_x^p)\omega^0 \wedge \omega^3 + \frac{1}{\sqrt{2}}(\varphi_x^p - \bar{\varphi}_x^p)\mathbf{m}^b \wedge \bar{\mathbf{m}}^b.$$

consistent with the general spherically symmetric expression for the gauge field in equation (14). A further computation shows that the associated Hodge dual  $\mathbf{F}^{p*}$  is given by

$$\mathbf{F}^{p*} = \frac{1}{\sqrt{2}}i(\bar{\varphi}_x^p - \varphi_x^p)\omega^0 \wedge \omega^3 - \frac{1}{\sqrt{2}}i(\bar{\varphi}_x^p + \varphi_x^p)\mathbf{m}^b \wedge \bar{\mathbf{m}}^b.$$

From the above expressions one can read the electric and magnetic parts of the spherically symmetric gauge field with respect to the normal of  $\mathcal{I}$ . One finds that

$$\mathbf{E}^p|_{\mathcal{I}} = -\frac{1}{\sqrt{2}}(\varphi_x^p + \bar{\varphi}_x^p)\omega^0, \quad \mathbf{B}^p|_{\mathcal{I}} = \frac{1}{\sqrt{2}}i(\varphi_x^p - \bar{\varphi}_x^p)\omega^0.$$

Thus, generically, both the electric and magnetic parts of the spherically symmetric gauge field given by equation (14)  $\mathbf{F}^p$  with respect to the normal of  $\mathcal{I}$  are non-zero as long as  $\varphi_{AB}^p$  is neither real nor pure imaginary. *Under this assumption, following the discussion of Section 3.5, the Poynting vector will be non-vanishing and hence there will be a flow of energy along the conformal boundary. Accordingly, the mass of the spacetime will not be constant.*

Again, for future reference define

$$\mathbf{u}_{SS} \equiv (e_x^0, e_x^3, e_z^+, e_y^-, f_x, \xi_{ex}, \chi_2, \chi_h, \theta_2, \theta_h, \theta_x, \phi_2, \varphi_x^p, A^p, A_x^p, A_y^p, A_z^p)$$

The components of the vector unknown  $\mathbf{u}$  not appearing in  $\mathbf{u}_{SS}$  will be denoted collectively by  $\mathbf{u}_{NSS}$ . *The above Ansatz is consistent with the evolution equations:* a lengthy computation using the suite `xAct` for tensorial and spinorial manipulations for `Mathematica`—see [18, 27]—shows

that the evolution equations implied by (31a)-(31b) for the components of  $\mathbf{u}_{NSS}$  are homogeneous in  $\mathbf{u}_{NSS}$ . Hence they admit the solution

$$\mathbf{u}_{NSS} = (0, \dots, 0).$$

Accordingly, components of the spinorial field not appearing in the Ansatz cannot appear in the course of the evolution if they have been set initially to zero. Taking into account the gauge conditions (30), the evolution equations for the components of  $\mathbf{u}_{SS}$  are given by

$$\partial_\tau e_x^0 = \frac{1}{3}(\chi_2 - \chi_h)e_x^0 - f_x, \quad (33a)$$

$$\partial_\tau e_x^3 = \frac{1}{3}(\chi_2 - \chi_h)e_x^3, \quad (33b)$$

$$\partial_\tau e_z^+ = -\frac{1}{6}(\chi_2 + 2\chi_h)e_z^+, \quad (33c)$$

$$\partial_\tau e_y^- = -\frac{1}{6}(\chi_2 + 2\chi_h)e_y^-, \quad (33d)$$

$$\partial_\tau f_x = \frac{1}{3}(\chi_2 - \chi_h)f_x + \theta_x, \quad (33e)$$

$$\partial_\tau \xi_x = -\frac{1}{6}(\chi_2 + 2\chi_h)\xi_x - \frac{1}{2}\chi_2 f_x - \theta_x, \quad (33f)$$

$$\partial_\tau \chi_2 = \frac{1}{6}(\chi_2 - 4\chi_h)\chi_2 - \theta_2 + \Theta\phi_2, \quad (33g)$$

$$\partial_\tau \chi_h = -\frac{1}{6}\chi_2^2 - \frac{1}{3}\chi_h^2 - \theta_h - \frac{3}{4}\Theta^2\delta_{pq}\varphi_x^p\bar{\varphi}_x^q, \quad (33h)$$

$$\partial_\tau \theta_x = \frac{1}{3}(\chi_2 - \chi_h)\theta_x - \frac{1}{3}d_x\phi_2 + \frac{1}{2}\Theta d_x\delta_{pq}\varphi_x^p\bar{\varphi}_x^q - \frac{1}{4}\Theta^2 f_x\delta_{pq}\varphi_x^p\bar{\varphi}_x^q, \quad (33i)$$

$$\begin{aligned} \partial_\tau \theta_2 = & \frac{1}{6}(\chi_2 - 2\chi_h)\theta_2 - \frac{1}{3}\chi_2\theta_h - \phi_2\dot{\Theta} + \frac{1}{4}\Theta^2\delta_{pq}\varphi_x^p\bar{\varphi}_x^q(3\chi_2 + 4\chi_h) \\ & - \Theta\dot{\Theta}\delta_{pq}\varphi_x^p\bar{\varphi}_x^q - \frac{1}{\sqrt{2}}\Theta^2\delta_{rs}\varphi_x^p\bar{\varphi}_x^r C^s{}_{pq}A^q - \frac{1}{\sqrt{2}}\Theta^2\delta_{rs}\varphi_x^r\bar{\varphi}_x^p\bar{C}^s{}_{pq}A^q, \end{aligned} \quad (33j)$$

$$\begin{aligned} \partial_\tau \theta_h = & -\frac{1}{6}\chi_2\theta_2 - \frac{1}{3}\chi_h\theta_h - \Theta\dot{\Theta}\delta_{pq}\varphi_x^p\bar{\varphi}_x^q + \frac{1}{4}\Theta^2\chi_h\delta_{pq}\varphi_x^p\bar{\varphi}_x^q \\ & - \frac{1}{4\sqrt{2}}\Theta^2\delta_{ps}\varphi_x^q\bar{\varphi}_x^p C^s{}_{qr}A^r - \frac{1}{4\sqrt{2}}\Theta^2\delta_{ps}\varphi_x^p\bar{\varphi}_x^q\bar{C}^s{}_{qr}A^r, \end{aligned} \quad (33k)$$

$$\begin{aligned} \partial_\tau \phi_2 = & -\frac{1}{2}(\chi_2 + 2\chi_h)\phi_2 + \frac{1}{2}(\chi_2 + 2\chi_h)\Theta\delta_{pq}\varphi_x^p\bar{\varphi}_x^q - \dot{\Theta}\delta_{pq}\varphi_x^p\bar{\varphi}_x^q \\ & - \frac{1}{\sqrt{2}}\Theta\delta_{qs}\varphi_x^p\bar{\varphi}_x^q C^s{}_{pr}A^r - \frac{1}{\sqrt{2}}\Theta\delta_{ps}\varphi_x^p\bar{\varphi}_x^q\bar{C}^s{}_{qr}A^r, \end{aligned} \quad (33l)$$

$$\partial_\tau \varphi_x^p = -\frac{1}{3}(\chi_2 + 2\chi_h)\varphi_x^p + \frac{1}{\sqrt{2}}C^p{}_{qr}\varphi_x^q A^r, \quad (33m)$$

$$\begin{aligned} \partial_\tau A^p - 2e_x^0\partial_\tau A_x^p - 2\sqrt{2}e_x^3\partial_3 A_x^p - \sqrt{2}e_z^+\partial_+ \alpha_z^p - \sqrt{2}e_y^-\partial_- \alpha_y^p \\ = 4\sqrt{2}\xi_x A_x^p - 2\sqrt{2}f_x A_x^p - 2\chi_h A^p + 2\sqrt{2}F^p(x), \end{aligned} \quad (33n)$$

$$\begin{aligned} \partial_\tau A_x^p - e_x^0\partial_\tau A^p - \sqrt{2}e_x^3\partial_3 A^p \\ = \frac{2}{3}(\chi_2 - \chi_h)A_x^p - \sqrt{2}(\varphi_x^p + \bar{\varphi}_x^p) - \sqrt{2}f_x A^p + \sqrt{2}C^p{}_{qr}A_x^q A^r, \end{aligned} \quad (33o)$$

$$\partial_\tau A_y^p - e_z^+\partial_+ A^p = -\frac{1}{3}(\chi_2 + 2\chi_h)A_y^p + \sqrt{2}C^p{}_{qr}A_y^q A^r \quad (33p)$$

$$\partial_\tau A_z^p - e_y^-\partial_- A^p = -\frac{1}{3}(\chi_2 + 2\chi_h)A_z^p + \sqrt{2}C^p{}_{qr}A_z^q A^r. \quad (33q)$$

In what follows, the above equations will be called the *spherically symmetric conformal evolution equations* for the Einstein-Yang-Mills system. Note that except for equations (33n)-(33q), the whole system consists of transport equations along conformal curves. A quick calculation shows that equations (33n)-(33q) can be rewritten as a symmetric hyperbolic subsystem for the components  $A^p$  and  $A_x^p$  of the gauge potentials. Equations (33i)-(33j) are derived from (8c) which contains derivatives of the energy-momentum tensor and hence of  $\varphi^p{}_{BC}$ . These derivatives obstruct the hyperbolicity of the evolution system. In [12] this problem has been dealt with by introducing an *auxiliary field*  $\psi^p{}_{AA'BC}$  describing the components of the derivative  $\nabla_{AA'}\varphi^p{}_{BC}$  and their corresponding evolution equations. This procedure considerably increases the complexity of the analysis. However, in spherical symmetry the derivative  $\nabla_{AA'}\varphi^p{}_{BC}$  has only two independent components —namely, those corresponding to the directions  $\mathbf{e}_0$  and  $\mathbf{e}_3$ . Moreover, in the derivation of the evolution equations only the time derivative  $\partial_\tau\varphi_x^p$  appears in the equation. This can be replaced by an expression not involving derivatives using the evolution equation (33m). Accordingly, it is possible to eliminate all terms involving derivatives of the gauge field spinor  $\varphi^p{}_{AB}$  in (33i)-(33j) and obtain their present form. Thus, the introduction of the auxiliary term  $\psi^p{}_{AA'BC}$  is not required and the overall analysis is considerably simplified in the spherically symmetric setting.

### 5.3.1 Initial data for the evolution equations

The conformal evolution equations (33a)-(33o) need to be supplemented with spherically symmetric initial data satisfying the conformal constraint equations on  $\mathcal{S}$ . The basic input for this construction is given by:

- (i) a *spherically symmetric* conformal factor  $\Omega$  on  $\mathcal{S}$  with

$$\Omega = 0, \quad d\Omega \neq 0, \quad \text{on } \partial\mathcal{S};$$

- (ii) 3-dimensional symmetric and *spherically symmetric* tensors  $\mathbf{h}$  and  $\mathbf{K}$  such that

$$\tilde{\mathbf{h}} = \Omega^{-2}\mathbf{h}, \quad \tilde{\mathbf{K}} = \Omega^{-1}\mathbf{K}$$

satisfy, on  $\tilde{\mathcal{S}} \equiv \mathcal{S} \setminus \partial\mathcal{S}$ , the *Hamiltonian* and *momentum constraints*

$$\tilde{D}^i \tilde{K}_{ij} - \tilde{D}_j \tilde{K}^j = \tilde{j}_j, \quad \tilde{r} - \tilde{K}^2 + \tilde{K}^{ij} \tilde{K}_{ij} = 2(\lambda - \tilde{\rho}),$$

with  $\tilde{r}$  and  $\tilde{D}$  denoting, respectively, the Ricci scalar and Levi-Civita connection of the physical metric  $\tilde{\mathbf{h}}$  and  $\tilde{\rho}$ ,  $\tilde{j}_j$  the *physical energy density* and *energy flux* of the Yang-Mills field;

- (iii) fields  $\varphi^{\mathbf{p}}$  prescribing the initial prescription of the gauge field strength which satisfy the Gauss constraint implied by the equations  $M^{\mathbf{p}}_{\mathbf{A}'\mathbf{A}} = 0$ , which, in the present setting take the form

$$e_x^3 \partial_3 \varphi_x^{\mathbf{p}} + 2\xi_x \varphi_x^{\mathbf{p}} - C^{\mathbf{p}}_{\mathbf{qr}} \varphi_x^{\mathbf{p}} A_x^{\mathbf{r}} = 0.$$

- (iv) an initial prescription of the gauge potential  $\mathbf{A}^{\mathbf{p}}$  satisfying the constraint implied by the equation  $M^{\mathbf{p}}_{\mathbf{AA}'\mathbf{BB}'} = 0$ , which, in the present setting takes the form

$$\begin{aligned} e_z^+ \partial_+ A_z^{\mathbf{p}} - e_y^- \partial_- A_y^{\mathbf{p}} + C^{\mathbf{p}}_{\mathbf{qr}} A_y^{\mathbf{q}} A_z^{\mathbf{r}} &= \sqrt{2}(\bar{\varphi}_x^{\mathbf{p}} - \varphi_x^{\mathbf{p}}), \\ e_z^+ \partial_+ A_x^{\mathbf{p}} - e_x^3 \partial_3 A_y^{\mathbf{p}} - \xi_x A_y^{\mathbf{p}} - C^{\mathbf{p}}_{\mathbf{qr}} A_x^{\mathbf{q}} A_y^{\mathbf{r}} &= 0, \\ e_y^- \partial_- A_x^{\mathbf{p}} - e_x^3 \partial_3 A_z^{\mathbf{p}} - \xi_x A_z^{\mathbf{p}} - C^{\mathbf{p}}_{\mathbf{qr}} A_x^{\mathbf{q}} A_z^{\mathbf{r}} &= 0. \end{aligned}$$

Given the above basic data, one can make use the conformal constraints on  $\mathcal{S}$  to compute the required initial data for the conformal evolution equations using the so-called *conformal constraint equations* —see [10, 12] for more details on this construction. The precise details of this construction will not be required in the subsequent discussion.

In addition to the observations made above, one has that in the present gauge

$$e_x^3 = 0, \quad f_x = 0, \quad \text{on } \partial\mathcal{S}.$$

## 5.4 Identifying the boundary conditions

In order to analyse the possible *maximally dissipative boundary conditions* associated to the equations (33n)-(33o) —see e.g. [16] for an introduction to the underlying theory— it is convenient to define the new variables

$$A_+^{\mathbf{p}} \equiv \frac{1}{2}(A_x^{\mathbf{p}} + A^{\mathbf{p}}), \quad A_-^{\mathbf{p}} \equiv \frac{1}{2}(A_x^{\mathbf{p}} - A^{\mathbf{p}}).$$

The new variables give rise to the following evolution subsystem

$$\begin{aligned} (1 - \frac{4}{3}e_x^0) \partial_\tau A_+^{\mathbf{p}} + \frac{1}{3} \partial_\tau A_-^{\mathbf{p}} - \frac{4\sqrt{2}}{3} e_x^3 \partial_3 A_+^{\mathbf{p}} &= \frac{4\sqrt{2}}{3} C^{\mathbf{p}}_{\mathbf{qr}} A_-^{\mathbf{q}} A_+^{\mathbf{r}} + \frac{4\sqrt{2}}{3} (A_+^{\mathbf{p}} + A_-^{\mathbf{p}}) \xi_x \\ &\quad - \frac{4\sqrt{2}}{3} A_+^{\mathbf{p}} f_x - \frac{4\sqrt{2}}{3} \varphi_x^{\mathbf{p}} + \frac{2\sqrt{2}}{3} \mathcal{F}^{\mathbf{p}} + \frac{2}{9} (A_-^{\mathbf{p}} - 5A_+^{\mathbf{p}}) \chi_h + \frac{4}{9} (A_+^{\mathbf{p}} + A_-^{\mathbf{p}}) \chi_2, \end{aligned} \quad (34a)$$

$$\partial_\tau A_y^{\mathbf{p}} - e_z^+ \partial_+ (A_+^{\mathbf{p}} - A_-^{\mathbf{p}}) = -\frac{1}{3} (\chi_2 + 2\chi_h) A_y^{\mathbf{p}} + \sqrt{2} C^{\mathbf{p}}_{\mathbf{qr}} A_y^{\mathbf{q}} (A_-^{\mathbf{r}} - A_+^{\mathbf{r}}) \quad (34b)$$

$$\partial_\tau A_z^{\mathbf{p}} - e_y^- \partial_- (A_+^{\mathbf{p}} - A_-^{\mathbf{p}}) = -\frac{1}{3} (\chi_2 + 2\chi_h) A_z^{\mathbf{p}} + \sqrt{2} C^{\mathbf{p}}_{\mathbf{qr}} A_z^{\mathbf{q}} (A_-^{\mathbf{r}} - A_+^{\mathbf{r}}), \quad (34c)$$

$$\begin{aligned} (1 + \frac{4}{3}e_x^0) \partial_\tau A_-^{\mathbf{p}} + \frac{1}{3} \partial_\tau A_+^{\mathbf{p}} + \frac{4\sqrt{2}}{3} e_x^3 \partial_3 A_-^{\mathbf{p}} &= \frac{4\sqrt{2}}{3} C^{\mathbf{p}}_{\mathbf{qr}} A_-^{\mathbf{q}} A_+^{\mathbf{r}} - \frac{4\sqrt{2}}{3} (A_+^{\mathbf{p}} + A_-^{\mathbf{p}}) \xi_x \\ &\quad + \frac{4\sqrt{2}}{3} A_-^{\mathbf{p}} f_x - \frac{4\sqrt{2}}{3} \varphi_x^{\mathbf{p}} - \frac{2\sqrt{2}}{3} \mathcal{F}^{\mathbf{p}} + \frac{2}{9} (A_+^{\mathbf{p}} - 5A_-^{\mathbf{p}}) \chi_h + \frac{4}{9} (A_+^{\mathbf{p}} + A_-^{\mathbf{p}}) \chi_2. \end{aligned} \quad (34d)$$

Notice that both equations contain, in their right hand sides, the gauge source functions  $\mathcal{F}^{\mathfrak{p}}$ . The system (34a)-(34d) can be written, schematically, in matrix form as

$$\mathbb{A}^0 \partial_\tau \mathbf{y} + \mathbb{A}^3 \partial_3 \mathbf{y} = \mathbb{B} \mathbf{y} \quad \text{where} \quad \mathbf{y} = \begin{pmatrix} A_+^{\mathfrak{p}} \\ A_y^{\mathfrak{p}} \\ A_z^{\mathfrak{p}} \\ A_-^{\mathfrak{p}} \end{pmatrix}.$$

Key to the identification of maximally dissipative boundary conditions is the *normal matrix* associated to this evolution subsystem—that is, the matrix associated to the  $\partial_3$  derivative evaluated at the boundary. A computation shows that it is given by

$$\mathbb{A}^3|_{\mathcal{S}^+} = \frac{4\sqrt{2}}{3} e^3 \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Maximally dissipative boundary conditions arise from the identification of the subspaces of  $\mathbb{R}^4$  for which the quadratic form associated to  $\mathbb{A}^3|_{\mathcal{S}^+}$  is non-positive. Setting for each  $\mathfrak{p}$  (note that here and in the following no summation is implied in the bundle index  $\mathfrak{p}$ )

$$A_-^{\mathfrak{p}} = c^{\mathfrak{p}} A_+^{\mathfrak{p}},$$

where  $c^{\mathfrak{p}} : \mathcal{S} \mapsto \mathbb{R}$  is a smooth functions on  $\mathcal{S}$ , one finds that

$$(A_+^{\mathfrak{p}}, A_y^{\mathfrak{p}}, A_z^{\mathfrak{p}}, A_-^{\mathfrak{p}}) \mathbb{A}^3 \begin{pmatrix} A_+^{\mathfrak{p}} \\ A_y^{\mathfrak{p}} \\ A_z^{\mathfrak{p}} \\ A_-^{\mathfrak{p}} \end{pmatrix} = \frac{4\sqrt{2}}{3} e^3 ((c^{\mathfrak{p}})^2 - 1) (A_+^{\mathfrak{p}})^2 \leq 0 \quad \text{if and only if} \quad |c^{\mathfrak{p}}| \leq 1.$$

Notice, in particular, that it is not possible to prescribe boundary conditions for the components  $A_y^{\mathfrak{p}}$  and  $A_z^{\mathfrak{p}}$ . More generally, one can consider *non-homogeneous maximally dissipative boundary conditions*

$$A_-^{\mathfrak{p}} = c^{\mathfrak{p}} A_+^{\mathfrak{p}} + q^{\mathfrak{p}}, \quad |c^{\mathfrak{p}}| \leq 1, \quad (35)$$

where  $c^{\mathfrak{p}}$  and  $q^{\mathfrak{p}}$  are smooth functions on  $\mathcal{S}$ . A discussion behind the motivation of this procedure to identify boundary conditions can be found in e.g. [16, 17].

In the following it will be assumed that spherically symmetric functions  $c^{\mathfrak{p}}$  and  $q^{\mathfrak{p}}$  have been specified on  $\mathcal{S}$ . As a result their values and derivatives  $\partial_\tau^n c^{\mathfrak{p}}$  and  $\partial_\tau^n q^{\mathfrak{p}}$  for  $n \in \mathbb{N}$  are known along  $\mathcal{S}$  and in particular on  $\partial\mathcal{S}$ . At this stage, it is important to point out that there is no guarantee that arbitrary choices of  $c^{\mathfrak{p}}$  and  $q^{\mathfrak{p}}$  give rise to smooth solutions of the field equations. As we will discuss in the next subsection, smoothness requires that the initial values for  $q^{\mathfrak{p}}$ ,  $A_+^{\mathfrak{p}}$  and  $A_-^{\mathfrak{p}}$  as well as their derivatives satisfy certain compatibility conditions at  $\partial\mathcal{S}$ .

#### 5.4.1 Corner conditions

In order to obtain smooth solutions to the initial boundary value problem under consideration, certain compatibility conditions between the initial data on  $\mathcal{S}$  and the boundary conditions  $\mathcal{S}$  need to be satisfied at  $\partial\mathcal{S}$ . These conditions are known as *corner conditions*.

The boundary condition (35) implies at  $\partial\mathcal{S}$  the condition

$$(q^{\mathfrak{p}})_{\otimes} = (A_-^{\mathfrak{p}})_{\otimes} - (c^{\mathfrak{p}})_{\otimes} (A_+^{\mathfrak{p}})_{\otimes}$$

where the subscript  $_{\otimes}$  denotes evaluation at  $\partial\mathcal{S}$ . Thus, the value of the function  $q^{\mathfrak{p}}$  at  $\partial\mathcal{S}$  is fixed by the values of  $(A_+^{\mathfrak{p}})_{\otimes}$  and  $(c^{\mathfrak{p}})_{\otimes}$ . If  $q^{\mathfrak{p}} = 0$ , then one sees that  $(A_+^{\mathfrak{p}})_{\otimes}$  and  $(A_-^{\mathfrak{p}})_{\otimes}$  cannot be prescribed independently of each other. The above is the first in a hierarchy of corner conditions.

A first order corner condition is obtained by considering the evolution equations for  $A_+^{\mathfrak{p}}$  and  $A_-^{\mathfrak{p}}$  and requiring consistency with the  $\partial_\tau$  derivative of the boundary condition (35). Direct



evaluation of the evolution equations (34a)-(34d) on  $\partial\mathcal{S}$ , using that  $e_x^0 = 0$  and  $e_x^3 = \sqrt{2}$  there, gives the equations

$$3(\partial_\tau A_+^{\mathbf{p}})_\otimes + (\partial_\tau A_-^{\mathbf{p}})_\otimes = 8(\partial_3 A_+^{\mathbf{p}})_\otimes + 4\sqrt{2}C^{\mathbf{p}}_{\mathbf{qr}}(A_-^{\mathbf{q}})_\otimes (A_+^{\mathbf{r}})_\otimes - 4\sqrt{2}(\varphi^{\mathbf{p}})_\otimes + 2\sqrt{2}(\mathcal{F}^{\mathbf{p}})_\otimes, \quad (36a)$$

$$3(\partial_\tau A_-^{\mathbf{p}})_\otimes + (\partial_\tau A_+^{\mathbf{p}})_\otimes = -8(\partial_3 A_-^{\mathbf{p}})_\otimes + 4\sqrt{2}C^{\mathbf{p}}_{\mathbf{qr}}(A_-^{\mathbf{q}})_\otimes (A_+^{\mathbf{r}})_\otimes - 4\sqrt{2}(\varphi^{\mathbf{p}})_\otimes - 2\sqrt{2}(\mathcal{F}^{\mathbf{p}})_\otimes. \quad (36b)$$

As the  $\partial_3$  derivative is intrinsic to the initial hypersurface, the values of  $(\partial_3 A_\pm^{\mathbf{p}})_\otimes$  can be computed from the initial data. Accordingly, equations (36a)-(36b) can be read as a linear algebraic system for  $(\partial_\tau A_\pm^{\mathbf{p}})_\otimes$ . The  $\partial_\tau$ -derivative of the boundary condition (35) then yields

$$(\partial_\tau q^{\mathbf{p}})_\otimes = (\partial_\tau A_-^{\mathbf{p}})_\otimes - (c^{\mathbf{p}})_\otimes (\partial_\tau A_+)_\otimes - (\partial_\tau c^{\mathbf{p}})_\otimes (A_+^{\mathbf{p}})_\otimes.$$

Consequently, substituting the value of  $(\partial_\tau A_\pm^{\mathbf{p}})_\otimes$  obtained from solving equations (36a)-(36b) one obtains an expression of the form

$$(\partial_\tau q^{\mathbf{p}})_\otimes = H[(A_\pm^{\mathbf{p}})_\otimes, (\partial_3 A_\pm^{\mathbf{p}})_\otimes, (\varphi^{\mathbf{p}})_\otimes, (F^{\mathbf{p}})_\otimes, (c^{\mathbf{p}})_\otimes, (\partial_\tau c^{\mathbf{p}})_\otimes].$$

Thus, the value of  $(\partial_\tau q^{\mathbf{p}})_\otimes$  is completely determined by the restriction of the initial data at  $\partial\mathcal{S}$ , the value of the gauge source function  $F^{\mathbf{p}}$  and  $c^{\mathbf{p}}$ . In the particular case of  $q^{\mathbf{p}} = 0$ , the above expression should be read as a constraint between  $(\partial_3 A_+^{\mathbf{p}})_\otimes$  and  $(\partial_3 A_-^{\mathbf{p}})_\otimes$ .

Higher corner conditions can be obtained, as necessary, by considering further  $\partial_\tau$ -derivatives of the evolution equations (34a)-(34d) and the boundary condition (35) and then evaluating these at  $\partial\mathcal{S}$ . Regarding derivatives of the form  $(\partial_3 \partial_\tau^n A_\pm^{\mathbf{p}})_\otimes$  computable from the initial data on  $\mathcal{S}$  and the (lower order) evolution equations, the  $n$ -th  $\partial_\tau$ -derivative of equations at  $\partial\mathcal{S}$  yields a linear algebraic system of equations for  $(\partial_\tau^{n+1} A_\pm^{\mathbf{p}})_\otimes$ . Substituting the result into the  $n+1$ -th  $\partial_\tau$ -derivative of the boundary condition (35) one obtains the value of  $(\partial_\tau^{n+1} q^{\mathbf{p}})_\otimes$ . Notice that this value will depend, among other things, on the value of the gauge source function  $F^{\mathbf{p}}$  and its  $\partial_\tau$ -derivatives at  $\partial\mathcal{S}$ .

The procedure described in the previous paragraphs shows that it is possible to construct, in a neighbourhood of  $\partial\mathcal{S}$  in  $\mathcal{I}$  a formal series expansion in  $\tau$  for the functions  $q^{\mathbf{p}}$ . Thus, the behaviour of the boundary data cannot be prescribed arbitrarily. In order to ensure smoothness of the solution to the boundary value problem in a neighbourhood  $\mathcal{W}$  of  $\partial\mathcal{S}$ , the general theory of initial boundary value problems for symmetric hyperbolic systems requires that the corner conditions described in the previous paragraphs are satisfied at every order. Notice that in concrete applications (for example a numerical simulation) it may only be feasible to impose the corner conditions to a finite order. The solution so obtained will be of class  $C^k$  for some  $k$  rather than  $C^\infty$ .

#### 5.4.2 The frame at the conformal boundary

The purpose of the present section is to show that the frame coefficient  $e_x^3$ , which appears in the normal matrix  $\mathbb{A}^3|_{\mathcal{I}^+}$ , can be determined purely from initial data on  $\partial\mathcal{S}$ . As a result the maximally dissipative boundary conditions (35) only prescribe the components  $A_\pm^{\mathbf{p}}$  of the gauge potential.

The key to this analysis is the observation that if  $\Theta = 0$ , the evolution equations (33a)-(33o) imply the following interior subsystem involving the coefficient  $e_x^3$ :

$$\partial_\tau e_x^3 = \frac{1}{3}(\chi_2 - \chi_h)e_x^3, \quad (37a)$$

$$\partial_\tau \chi_2 = \frac{1}{6}(\chi_2 - 4\chi_h)\chi_h - \theta_2, \quad (37b)$$

$$\partial_\tau \chi_h = -\frac{1}{6}\chi_2^2 - \frac{1}{3}\chi_h^2 - \theta_h, \quad (37c)$$

$$\partial_\tau \theta_2 = \frac{1}{6}(\chi_2 - 2\chi_h)\theta_2 - \frac{1}{3}\chi_2\theta_h, \quad (37d)$$

$$\partial_\tau \theta_h = -\frac{1}{6}\chi_2\theta_2 - \frac{1}{3}\chi_h\theta_h. \quad (37e)$$

Initial data for the above system is obtained by recalling that in the present gauge

$$\Sigma = 0, \quad s = 0, \quad \text{on} \quad \partial\mathcal{S}$$

so that, in particular,

$$\chi_2 = 0, \quad \chi_h = 0 \quad \text{on} \quad \partial\mathcal{S}. \quad (38)$$

Now, as  $s = \Omega\varsigma_\star$ , it follows from the conformal constraint equation (15c), using  $e_3(\Omega)|_{\partial\mathcal{S}} \neq 0$ , that

$$L_{33} = \varsigma_\star, \quad \text{on} \quad \partial\mathcal{S}.$$

As a consequence of the spherical symmetry, the above is the only non-vanishing component of  $L_{33}$ . Accordingly, one finds that

$$L_{(\mathbf{AB})(\mathbf{CD})} = \varsigma_\star x_{\mathbf{AB}} x_{\mathbf{CD}}.$$

Thus, using that

$$x_{(\mathbf{AB})x(\mathbf{CD})} = 2\epsilon_{\mathbf{ABCD}}^2, \quad h_{\mathbf{ABCD}} x^{\mathbf{AB}} x^{\mathbf{CD}} = 1,$$

one obtains that

$$\theta_2 = 2\varsigma_\star, \quad \theta_h = -\varsigma_\star \quad \text{on} \quad \partial\mathcal{S}. \quad (39)$$

Finally, we observe that  $e_{01}$  is chosen to give the unit normal  $\mathbf{N}$  of  $\mathcal{S}$  at  $\partial\mathcal{S}$  and that

$$e_x^3 = \sqrt{2}, \quad \text{on} \quad \partial\mathcal{S}. \quad (40)$$

Using the initial data (38), (39) and (40), it can be verified that the solution to the interior subsystem (37a)-(37e) is given by

$$\begin{aligned} e_x^3 &= \frac{\sqrt{2}}{1 + \frac{1}{2}\varsigma_\star\tau^2}, \\ \chi_2 &= -\frac{2\varsigma_\star\tau}{1 + \frac{1}{2}\varsigma_\star\tau^2}, \\ \chi_h &= \frac{\varsigma_\star\tau}{1 + \frac{1}{2}\varsigma_\star\tau^2}, \\ \theta_2 &= \frac{2\varsigma_\star}{1 + \frac{1}{2}\varsigma_\star\tau^2}, \\ \theta_h &= -\frac{\varsigma_\star}{1 + \frac{1}{2}\varsigma_\star\tau^2}. \end{aligned}$$

The previous analysis can be summarised as follows:

**Lemma 2.** *For any solution to the conformal evolution equations (33a)-(33o) satisfying the conformal constraint equations on  $\partial\mathcal{S}$  one has, irrespective of the values taken by the gauge potential  $A^{\mathbf{p}}_{\mathbf{AA}'}$  on  $\mathcal{S}$ , that the normal matrix is given by*

$$\mathbb{A}^3|_{\mathcal{S}^+} = \frac{16}{6 + 3\varsigma_\star\tau^2} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus, the maximally dissipative boundary conditions (35) only imply conditions on the gauge potentials.

## 5.5 Propagation of the constraints

In order to conclude the construction of solutions to the conformal Einstein-Yang-Mills field equations, it is necessary to provide a discussion of the so-called *propagation of the constraints*. This analysis requires the construction of a suitable subsidiary evolution system for the zero-quantities representing the various conformal field equations. In addition, it is necessary to consider subsidiary equations for the zero-quantities

$$\begin{aligned} \delta_a &\equiv b_a - f_a - \Theta^{-1} \hat{\nabla}_a \Theta, \\ \gamma_{ab} &\equiv \frac{1}{2} \Theta^2 T_{ab} + \frac{1}{6} \Theta^{-2} \eta_{ab} - \hat{L}_{ab} - \hat{\nabla}_a b_b - \frac{1}{2} S_{ab}{}^{cd} b_c b_d, \\ \varsigma_{ab} &\equiv \hat{L}_{ab} - \hat{\nabla}_{[a} f_{b]}, \end{aligned}$$

associated to the conformal gauge used in the hyperbolic reduction of the conformal field equations.

In order to construct the required subsidiary evolution system we follow the procedure discussed in [13] for the extended vacuum conformal field equations and adapt it, as necessary, to the particular features of the Yang-Mills equations —see e.g. [12]. A particular case of this analysis has been carried out in [25] where the extended conformal Einstein-Maxwell system was considered. In what follows, we concentrate on the structural properties of this computationally intensive analysis. In particular, for the sake of conciseness, whenever possible we make use of the tensorial counterpart of the equations.

Assuming that the evolution equations (31a)-(31b) hold, a lengthy computation shows that

$$\partial_\tau \delta_a = H_a[\delta_a, \gamma_{ab}, \varsigma_{ab}, \hat{\Sigma}_a^b{}_c], \quad (41a)$$

$$\partial_\tau \gamma_{ab} = H_{ab}[\gamma_{ab}], \quad (41b)$$

$$\partial_\tau \varsigma_{ab} = H_{ab}[\hat{\Xi}^c{}_{dab}], \quad (41c)$$

where the terms  $H[\dots]$  in the right hand side of the equations denote expressions which are homogeneous in the zero-quantities appearing in brackets. A further computation shows that for the geometric zero-quantities  $\hat{\Sigma}_a^b{}_c$ ,  $\hat{\Xi}^c{}_{dab}$  and  $\hat{\Delta}_{cab}$  one has subsidiary equations of the form

$$\partial_\tau \hat{\Sigma}_a^b{}_c = H_a^b{}_c[\hat{\Sigma}_a^b{}_c, \hat{\Xi}^c{}_{dab}], \quad (42a)$$

$$\partial_\tau \hat{\Xi}^c{}_{dab} = H^c{}_{dab}[\hat{\Sigma}_a^b{}_c, \hat{\Xi}^c{}_{dab}, \hat{\Delta}_{cab}, \Lambda_{cab}], \quad (42b)$$

$$\partial_\tau \hat{\Delta}_{cab} = H_{cab}[\hat{\Sigma}_a^b{}_c, \hat{\Delta}_{cab}, \Lambda_{cab}, M^p{}_a, M^{p*}{}_a]. \quad (42c)$$

For the matter constraint  $M^p{}_Q{}^Q$  a direct computation assuming the evolution equations (31a)-(31b) yields an equation of the form

$$\partial_\tau M^p{}_Q{}^Q = H[M^p{}_Q{}^Q, \hat{\Xi}^c{}_{dab}], \quad (43)$$

while the analysis of [12] shows that the zero-quantity  $M^p{}_{ab}$  has a subsidiary equation of the form

$$\partial_\tau M^p{}_{ab} = H[M^p{}_{ab}, M^p{}_a, M^{p*}{}_a, \hat{\Sigma}_a^b{}_c, \hat{\Xi}^c{}_{dab}]. \quad (44)$$

Finally, for  $Q^p \equiv \nabla^{PQ} A^p_{PQ} - F^p$ , one finds a subsidiary system of the form

$$\partial_\tau Q^p = H^p[Q^p, M^p{}_{ab}, M^p{}_a, M^{p*}{}_a, \hat{\Sigma}_a^b{}_c, \hat{\Xi}^c{}_{dab}] \quad (45)$$

Besides their homogeneity in the zero-quantities, the key feature of the subsidiary equations (41a)-(41c), (42a)-(42c), (43), (44) and (45) is that they are all transport equations. Accordingly, they do not have to be supplemented by a boundary condition.

The analysis of the subsidiary equation associated to the zero-quantity  $\Lambda_{cab}$  is much more delicate. Following the strategy discussed in [13], the boundary adapted Bianchi evolution system implies a subsidiary equation system of the form

$$\partial_\tau C_{00} + e_{00}^\mu \partial_\mu C_{01} = U_{00}[\hat{\Sigma}_a^b{}_c, \hat{\Xi}^c{}_{dab}, \varsigma_{ab}, M^p{}_a, M^{p*}{}_a], \quad (46a)$$

$$\partial_\tau C_{01} + e_{00}^\mu \partial_\mu C_{11} - e_{11}^\mu \partial_\mu C_{00} = U_{01}[\hat{\Sigma}_a^b{}_c, \hat{\Xi}^c{}_{dab}, \varsigma_{ab}, M^p{}_a, M^{p*}{}_a], \quad (46b)$$

$$\partial_\tau C_{11} - e_{11}^\mu \partial_\mu C_{01} = U_{11}[\hat{\Sigma}_a^b{}_c, \hat{\Xi}^c{}_{dab}, \varsigma_{ab}, M^p{}_a, M^{p*}{}_a], \quad (46c)$$

for the components  $C_{AB}$  of the Bianchi constraints. It can be readily verified that the above system has a vanishing normal matrix. Accordingly, the subsidiary equations (46a)-(46c) do not give rise to boundary conditions.

From the homogeneity in the zero quantities of the subsidiary equations (41a)-(41c), (42a)-(42c), (43), (44) and (46a)-(46c), and the absence of further boundary conditions we readily obtain the following *reduction lemma*:

**Lemma 3** (Reduction Lemma). *Let  $p \in \partial\mathcal{S}$ , where  $\mathcal{U}$  is an open neighbourhood of  $p$  in  $[0, \infty) \times \mathcal{S}$  and  $\mathcal{V} \equiv \mathcal{U} \cap (\mathcal{S} \cup \mathcal{I})$ . Assume one has a smooth solution to the conformal evolution equations (31a)-(31b) for data on  $\mathcal{V}$  in the boundary adapted gauge which satisfy on  $\mathcal{V} \cap \mathcal{S}$  the conformal constraint equations. Finally, denote by  $\mathbf{g}$  the metric obtained from the orthonormal frame  $\mathbf{e}_a$  and by  $D^+(\mathcal{V})$  the future domain of dependence of  $\mathcal{V}$  in  $\mathcal{U}$  with respect to  $\mathbf{g}$ . Then the extended conformal Einstein-Yang-Mills equations are satisfied on  $D^+(\mathcal{V})$  by the conformal fields solving the conformal evolution equations (31a)-(31b).*

## 6 Main result

Given a hypersurface  $\mathcal{S}$  with boundary  $\partial\mathcal{S}$ , the natural domain to look for solutions to an initial boundary value problem for the conformal Einstein-Yang-Mills equations (33a)-(33o) with initial data prescribed on  $\mathcal{S}$  and boundary conditions on  $[0, \infty) \times \partial\mathcal{S}$  is of the form  $[0, \infty) \times \mathcal{S}$ . Using the theory of initial boundary value problems for symmetric hyperbolic systems with maximally dissipative boundary conditions as described in [19, 30] one has the following existence theorem for solutions to the conformal Einstein-Yang-Mills system:

**Theorem 1.** *Given spherically symmetric initial data  $\mathbf{u}_\star$  for the conformal Einstein-Yang-Mills field equations with an anti-de Sitter-like cosmological constant on an initial hypersurface  $\mathcal{S}$ , smooth gauge source functions  $\mathcal{F}^p$  on  $[0, \infty) \times \mathcal{S}$ , and smooth functions  $c^p$  and  $q^p$  on  $[0, \infty) \times \partial\mathcal{S}$  with  $|c^p| \leq 1$  satisfying the required corner conditions to any order, there exists for some  $T > 0$  a unique solution  $\mathbf{u}$  of the conformal field equations on a domain*

$$\mathcal{M}_T \equiv \{p \in [0, \infty) \times \mathcal{S} \mid 0 \leq \tau(p) \leq T\}$$

such that

$$\mathbf{u}|_{\mathcal{S}} = \mathbf{u}_\star, \quad (A_-^p - c^p A_+^p)|_{\mathcal{I} \cap \mathcal{M}_T} = q^p,$$

and

$$\nabla^a A^p_a = \mathcal{F}^p \quad \text{on } \mathcal{M}_T.$$

Moreover, the fields  $(\tilde{\mathbf{g}}, \mathbf{F}^p, \mathbf{A}^p)$  obtained from the solution  $\mathbf{u}$  to the conformal Einstein-Yang-Mills equations constitute a solution to the Einstein-Yang-Mills system on  $\tilde{\mathcal{M}}_T \equiv \mathcal{M}_T \setminus \mathcal{I}$  for which  $\mathcal{I}$  represents null infinity.

The above result constitutes an alternative formulation of the main theorem stated in the introductory section.

*Proof.* The existence of solutions to the conformal evolution system (33a)-(33o) follows from the assumptions on the initial data, boundary data and the corner conditions using the general theory of initial boundary value problems with maximally dissipative boundary data as given in [19, 30] and applied in [13, 16]. Once a solution to the reduced system is obtained, a solution to the full conformal Einstein-Yang-Mills equations follows by the assumption on the initial data and the *Reduction Lemma*. From here, general properties of the conformal Einstein field equations imply the existence of a solution to the Einstein-Yang-Mills equations away from the conformal boundary —see [10].  $\square$

## 7 Concluding remarks

The purpose of this article has been the formulation of an initial boundary value problem for the Einstein-Yang-Mills equations in a conformal setting which allows to show the local existence of a big class spherically symmetric solutions to these equations which behave, asymptotically, like the anti-de Sitter spacetime. The use of conformal methods allows to identify a great class of boundary conditions ensuring the well-posedness of the problem.

The present analysis is a first natural step towards a formulation of the local existence of non-symmetric anti-de Sitter like solutions to the Einstein-Yang-Mills equations. As a consequence of the spherical symmetry, no boundary data involving the geometric variables can be prescribed.

This situation is bound to change in the general, non-symmetric, setting where intuitively one would expect to be able to prescribe a linear combination of components of the Weyl tensor (expressed in terms of a boundary adapted frame). On the matter side, in addition to the boundary conditions for the gauge potential 1-form, boundary conditions for the gauge field will be required. The details of this intricate construction will be discussed elsewhere.

We expect the spherically symmetric conformal evolution system (33a)-(33o) to be amenable to a numerical implementation. The simulations obtained from such implementation will give valuable information concerning the global existence and stability of the local solutions constructed in the present work.

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## Appendix

Any symmetric rank 2 spinorial field  $X_{AB} = X_{(AB)}$  can be decomposed in terms of the *basic spinors*

$$x_{AB} \equiv \sqrt{2}\delta_{(A}^0\delta_{B)}^1, \quad y_{AB} \equiv -\frac{1}{\sqrt{2}}\delta_A^1\delta_B^1, \quad z_{AB} \equiv \frac{1}{\sqrt{2}}\delta_A^0\delta_B^0.$$

From the above, only  $x_{AB}$  has spin-weight 0. Similarly, any totally symmetric rank 4 spinorial field  $X_{ABCD}$  can be constructed using the basic spinors

$$\begin{aligned} \epsilon_{ABCD}^0 &\equiv \delta_{(A}^0\delta_B^0\delta_C^0\delta_{D)}^0, & \epsilon_{ABCD}^1 &\equiv \delta_{(A}^0\delta_B^0\delta_C^0\delta_{D)}^1, & \epsilon_{ABCD}^2 &\equiv \delta_{(A}^0\delta_B^0\delta_C^1\delta_{D)}^1 \\ \epsilon_{ABCD}^3 &\equiv \delta_{(A}^0\delta_B^1\delta_C^1\delta_{D)}^1, & \epsilon_{ABCD}^4 &\equiv \delta_{(A}^1\delta_B^1\delta_C^1\delta_{D)}^1. \end{aligned}$$

More general rank 4 spinors with the pairwise symmetry  $X_{ABCD} = X_{(AB)(CD)}$  are constructed using the above and the combinations

$$\begin{aligned} x_{AC}\epsilon_{BD} + x_{BD}\epsilon_{AC}, & \quad y_{AC}\epsilon_{BD} + y_{BD}\epsilon_{AC}, & z_{AC}\epsilon_{BD} + z_{BD}\epsilon_{AC}, \\ h_{ABCD} &\equiv -\epsilon_{A(C}\epsilon_{D)B}. \end{aligned}$$

A number of identities for the above objects can be found in [15]. It is noticed that only  $\epsilon_{ABCD}^2$ ,  $x_{AC}\epsilon_{BD} + x_{BD}\epsilon_{AC}$  and  $h_{ABCD}$  have spin weight 0.

## References

- [1] A. Ashtekar & A. Magnon, *Asymptotically anti-de Sitter space-times*, Class. Quantum Grav. **1**, L39 (1984).
- [2] R. Bartnik & J. McKinnon, *Particlelike Solutions of the Einstein- Yang-Mills Equations*, Phys. Rev. Lett. **61**, 141 (1988).
- [3] P. Bizon, *Colored black holes*, Phys. Rev. Lett. **64**, 2844 (1990).
- [4] P. Bizon, *Is AdS stable?*, In [arXiv:1312.5544](#), 2013.
- [5] P. Bizon & A. Rostworowski, *Weakly turbulent instability of anti-de Sitter spacetime*, Phys. Rev. Lett. **107**, 031102 (2011).
- [6] P. Breitenlohner, P. Forgács, & D. Maison, *Static Spherically Symmetric Solutions of the Einstein-Yang-Mills Equations*, Comm. Math. Phys. **163**, 141 (1994).
- [7] M. W. Choptuik, T. Chmaj, & P. Bizon, *Critical Behavior in Gravitational Collapse of a Yang-Mills Field*, Phys. Rev. Lett. **77**, 424 (1996).

- [8] J. Ehlers, *Spherically symmetric spacetimes*, in *Relativity, Astrophysics and Cosmology*, edited by W. Israel, page 114, D. Reidel Publishing Company, 1973.
- [9] P. Forgács & N. S. Manton, *Space-Time Symmetries in Gauge Theories*, Comm. Math. Phys. **72**, 15 (1980).
- [10] H. Friedrich, *Cauchy problems for the conformal vacuum field equations in General Relativity*, Comm. Math. Phys. **91**, 445 (1983).
- [11] H. Friedrich, *On the hyperbolicity of Einstein's and other gauge field equations*, Comm. Math. Phys. **100**, 525 (1985).
- [12] H. Friedrich, *On the global existence and the asymptotic behaviour of solutions to the Einstein-Maxwell-Yang-Mills equations*, J. Diff. geom. **34**, 275 (1991).
- [13] H. Friedrich, *Einstein equations and conformal structure: existence of anti-de Sitter-type space-times*, J. Geom. Phys. **17**, 125 (1995).
- [14] H. Friedrich, *Gravitational fields near space-like and null infinity*, J. Geom. Phys. **24**, 83 (1998).
- [15] H. Friedrich & J. Kánnár, *Bondi-type systems near space-like infinity and the calculation of the NP-constants*, J. Math. Phys. **41**, 2195 (2000).
- [16] H. Friedrich & G. Nagy, *The Initial Boundary Value Problem for Einstein's Vacuum Field Equation*, Comm. Math. Phys. **201**, 619 (1999).
- [17] H. Friedrich & A. D. Rendall, *The Cauchy problem for the Einstein equations*, Lect. Notes. Phys. **540**, 127 (2000).
- [18] A. García-Parrado & J. M. Martín-García, *Spinors: a Mathematica package for doing spinor calculus in General Relativity*, Comp. Phys. Commun. **183**, 2214 (2012).
- [19] O. Guès, *Problème mixte hyperbolique quasi-linéaire caractéristique*, Comm. Part. Diff. Eqns. **15**, 595 (1990).
- [20] G. Holzegel & J. Smulevici, *Self-gravitating Klein-Gordon fields in asymptotically anti de Sitter spacetimes*, Ann. Henri Poincaré **13**, 991 (2012).
- [21] G. Holzegel & J. Smulevici, *Stability of Schwarzschild-AdS for the spherically symmetric Einstein-Klein-Gordon system*, Comm. Math. Phys. **317**, 205 (2013).
- [22] G. Holzegel & C. M. Warnick, *The Einstein-Klein-Gordon-AdS system for general boundary conditions*, In [arXiv:1312.5332](https://arxiv.org/abs/1312.5332), 2013.
- [23] H. P. Künzle, *SU(n)-Einstein-Yang-Mills fields with spherical symmetry*, Class. Quantum Grav. **8**, 2283 (1991).
- [24] H. P. Künzle & A. K. M. M. ul Alam, *Spherically symmetric static SU(2) Einstein-Yang-Mills fields*, J. Math. Phys. **31**, 928 (1990).
- [25] C. Lübbe & J. A. Valiente Kroon, *The extended Conformal Einstein field equations with matter: the Einstein-Maxwell system*, J. Geom. Phys. **62**, 1548 (2012).
- [26] M. Maliborski & A. Rostworowski, *Lecture Notes on Turbulent Instability of Anti-de Sitter Spacetime*, J. Mod. Phys. A **28**, 1340020 (2013).
- [27] J. M. Martín-García, <http://metric.iem.csic.es/Martin-Garcia/xAct/>.
- [28] R. Penrose & W. Rindler, *Spinors and space-time. Volume 1. Two-spinor calculus and relativistic fields*, Cambridge University Press, 1984.

- [29] R. Penrose & W. Rindler, *Spinors and space-time. Volume 2. Spinor and twistor methods in space-time geometry*, Cambridge University Press, 1986.
- [30] J. Rauch, *Symmetric positive systems with boundary characteristic of constant multiplicity*, Trans. Am. Math. Soc. **291**, 167 (1985).
- [31] O. Rinne & V. Moncrief, *Hyperboloidal Einstein-matter evolution and tails for scalar and Yang-Mills fields*, Class. Quantum Grav. **30**, 095009 (2013).
- [32] J. Stewart, *Advanced general relativity*, Cambridge University Press, 1991.
- [33] J. A. Valiente Kroon, *Global evaluations of static black hole spacetimes*, In preparation.
- [34] R. M. Wald, *General Relativity*, The University of Chicago Press, 1984.
- [35] E. Witten, *Some Exact Multipseudoparticle Solutions of Classical Yang-Mills Theory*, Phys. Rev. Lett. **38**, 121 (1977).